CHAPTER 2

II. Integral conservation laws

The fundamental law of dynamics can be written as:

$$\frac{d(m\vec{V})}{dt} = \sum \vec{F}_{ext}$$

$$\sum \vec{F}_{ext} = \text{forces de volume} + \text{forces de surface}$$

The forces of volume are the forces of gravity

Surface forces are **pressure forces** + **friction forces** (shear forces).

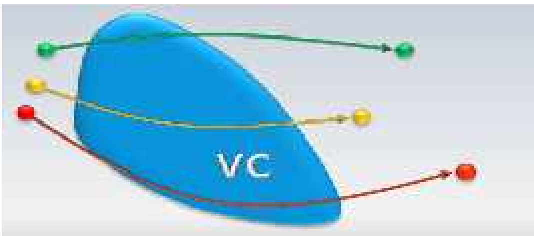
Shear forces are assumed to be negligible, and only inertia and pressure forces are associated with motion.

And so $\frac{d(m\vec{V})}{dt} = \sum \vec{F}_{ext} = \text{forces de gravité} + \text{forces de pression}$

- We'll apply Reynolds' transport theorem to successively find the **conservation** equations **for mass, momentum and energy.**
- Simplify the conservation equations for one-dimensional steady-state flow in a non-viscous fluid.
- The evolution of physical quantities (mass, momentum, energy) is analyzed using **integral balance equations** on macroscopic domains. It is therefore necessary to establish a **correspondence between a balance sheet and the transport of** physical quantities by flow.

II.1 Control volume V(C)

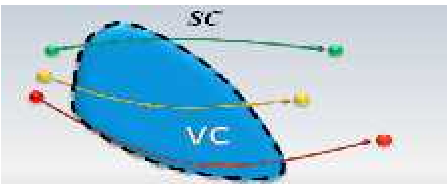
A control volume is an imaginary volume through which fluid can flow. The focus is on the physical quantities passing through the surface.



II.2 Control surface S(C)

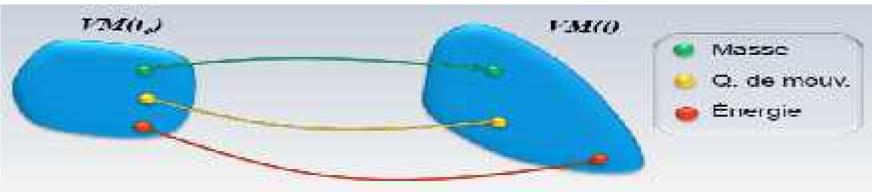
The envelope of a control volume is called the control surface. It can be fixed or mobile.

A control volume is an open system



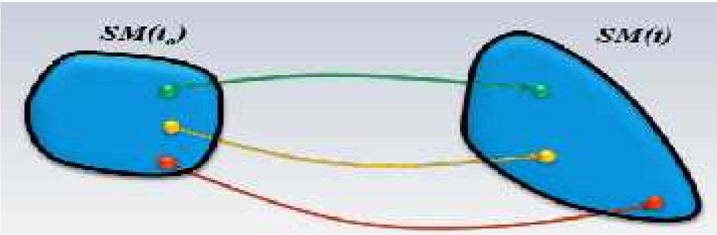
II.3 Material volume V(M)

A material volume is a portion of fluid that moves and deforms, but remains made up of the same set of particles.



II.4 Material surface S(M)

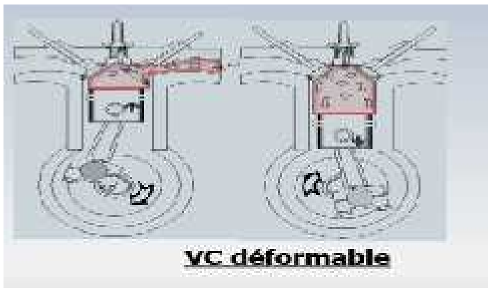
The envelope of a material volume is called the material surface. Each point of this envelope is a fluid particle.



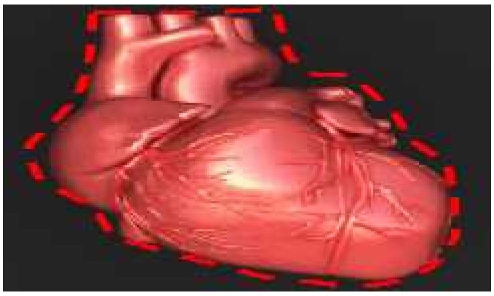
A material volume is a closed system!

The closed system (*VM*) is associated with Lagrangian kinematics, while the open system (*VC*) with the Eulerian approach.

Although the control volume (*CV*) is *often* fixed, it can also move or deform.



V(C) deformable



II.5 Reynolds transport theorem

Before looking at the ***Reynolds Transport Theorem,*** we need to introduce a few basic concepts.

II.5.1 Flow concept

To measure the quantity of matter passing through a surface (*S)* per unit of time and surface area, we introduce the notion of **flow:** *flow of mass, momentum, energy, etc.*

- **Volume and mass flow**

The elementary volume flow *dq* through a surface ***dS*** is the volume of fluid ***dv*** that passes through this surface in a time interval ***dt***, i.e.:

$$q_v = \oint_s \vec{V}.\vec{n}ds \qquad q_v = \frac{dv}{dt} = \vec{V}.d\vec{s} = \vec{V}.\vec{n}.ds$$

$$q_v = \int \vec{V}.\vec{n}.ds$$

Then, the total flow Q over a surface S is

Likewise for mass flow: $q = q_m \partial_v$ or again $q_m = \int \partial \vec{V}.\vec{n}.ds$

- **Intensive quantities**

An **extensive quantity** is proportional to the mass of the system**.**

We therefore have the following relationships : $m = \int_v \partial dv$

$$q_m = \frac{dm}{dt} = \int \partial \vec{V}.\vec{n}.ds$$

Mass ***m is*** an **extensive quantity, as** are the other two quantities of interest in mechanics, namely momentum ***m.***

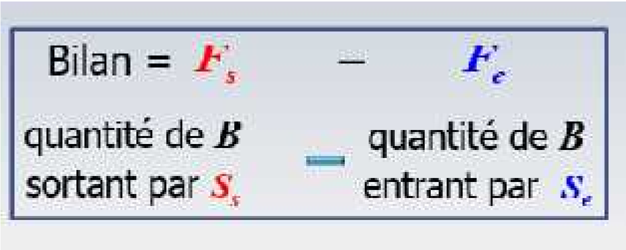
- **Extensive quantities**

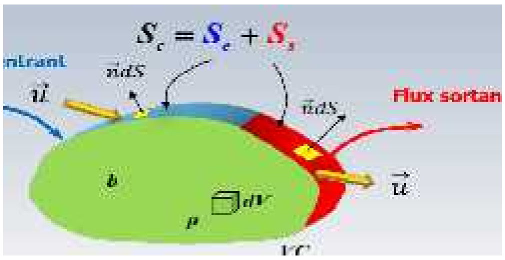
Intensive quantities, independent of the mass of the system. In particular, we'll be looking at quantities associated with mass *m*, momentum *mV and energy E.*
To do this, we reduce each of the properties (*m, mV,E)* by the mass m, to obtain the intensive quantities (1,V,e).
These relationships can be generalized for any *extensive quantity B* with a corresponding *intensive quantity,* i.e. *per unit mass, b = B /m*:

$$B = \int_v \rho b\,dv \qquad \dot{B} = \frac{dB}{dt} = \int_s \rho b\vec{V}ds$$

- **Net flow from B**

Consider a volume *VC* (fixed) bounded by *Sc*, through which flows a fluid carrying *B*



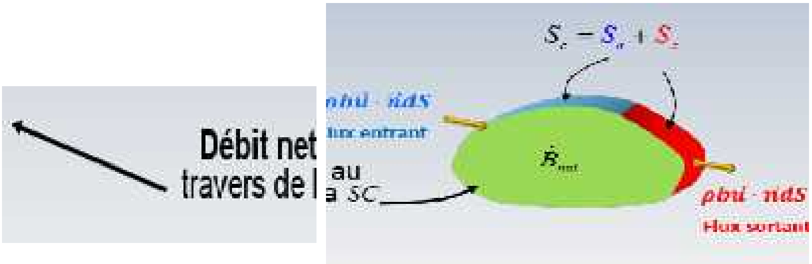


The net flow of ***B*** across the control surface (*SC*) can therefore be written as :

$$\dot{B} = \frac{dB}{dt} = \int_{Ss} \rho b\vec{V}ds - \int_{Se} \rho b\vec{V}ds$$

$$\dot{B}_{net} = \int_{Sc} \rho b\vec{V}ds$$

with $b = B/m$:



The flow rate $\dot{B}_{net}$ across the surface of a control volume, corresponds to the quantity of B that "accumulates" (negative or positive) per unit of time in the control volume. This variation in B in the control volume can be written:

$$\left(\frac{dB}{dt}\right)_{Vc} = \frac{d}{dt}\int_{Vc} \rho b\,dv$$

Accumulation over time in the control volume.

- **Assessment of a control volume**

For a control volume, in the absence of sources (wells), we recognize the following principle:

Accumulation (in V(c) + balance of flows(through S(c)=0

In mathematical form, we write the above principle as.

$$\underbrace{\frac{d}{dt}\int_{VC} \rho b dV}_{\text{Variation de } B \text{ dans le volume de contrôle}} + \underbrace{\int_{SC} \rho b(\vec{u}\cdot\vec{n})dS}_{\text{Flux de } B \text{ au travers de la surface de contrôle}} = 0$$

Special case for mass

In the case of conservation of mass, for example, the general equation.

$$\frac{d}{dt}\int_{Vc}\rho b dv + \int_{Sc}\rho b\vec{V}.\vec{n}ds = 0$$

For b=1

$$\frac{d}{dt}\int_{Vc}\rho dv + \int_{Sc}\rho\vec{V}.\vec{n}ds = 0$$

Mass **accumulation** + **Mass flow** through surface
in volume over time

$\left[\frac{\text{masse}}{\text{temps}}\right]$ $\left[\frac{\text{masse}}{\text{temps}}\right]$

On the material (moving) volume ***VM,*** we can look at the entire quantity ***B*** as $B = \int_{VM(t)}\rho b dv$

To obtain the temporal variation of ***B***, we need to calculate: $\frac{dB}{dt} = \frac{d}{dt}\int_{VM(t)}\rho b dv$

This evaluation has one drawback, since ***VM*** is also a function of time! In other words, the system is **moving and deforming: VM(t+dt)$\neq$ VM(t)**

$$\frac{dB}{dt} = \frac{d}{dt}\int_{Vc}\rho b dv + \int_{Sc}\rho b(\vec{V}.\vec{n})ds = 0$$

Time variation of **B** when following the system	=	Time variation of **B** when following the system	+	Net flow of **B** through surface Sc of volume Vc

II.5.2 Formulation of the *Reynolds Transport Theorem, TTR*

$$\left.\frac{dB}{dt}\right|_{\text{système}} = \frac{d}{dt}\int_{Vc}\rho b dv + \int_{Sc}\rho b(\vec{V}.\vec{n})ds = 0$$

This relationship was presented for a fixed control volume.

If the control volume deforms, consider *the relative velocity* $\vec{V}_{rel} = \vec{V}_{fluid} - \vec{V}_s$ between the velocity V of the fluid and that Vs of the volume Vc.

$$\boxed{\left.\frac{dB}{dt}\right|_{\text{système}} = \frac{d}{dt}\int_{Vc}\rho b\,dv + \int_{Sc}\rho b(\vec{V}_{rel}.\vec{n})ds = 0}$$ with $\vec{V}_{rel} = \vec{V}_{fluid} - \vec{V}_{s}$, **Vs=0 if Vc is fixed**

B	b=dB/dm
Mass m	1
Q.mouv mV	V
Energy E	e

B: extensive property. A quantity in the closed system
B: intensive property. Property *B* per unit mass
ρ Density of the fluid
$\vec{V}$: fluid velocity
$\vec{V}_{rel} = \vec{V}$:if the surface of the control volume is fixed
$\vec{V}_{rel} = \vec{V} - \vec{V}_{s}$ if the surface of the control volume is moving at speed $\vec{V}_{s}$ (like a beating heart)
dS: elementary area on the control surface, *Sc*
$\vec{n}$ outward unit normal of the elementary *SC dS*
dv: volume element in the Vc

II.6 Conservation of mass

In this case, let's analyze the conservation of mass with B=m and b=B/m=1.
Even if the material volume deforms, the mass in it remains the same over time.

$$\left.\frac{dB}{dt}\right|_{\text{système}} = \left.\frac{dm}{dt}\right|_{\text{système}} = q_m = 0$$

(The mass of a system remains constant over time)
Since there is no accumulation (or loss) of mass takes place in the control volume. The sum of positive and negative flows (volumes) is zero.
The conservation of mass equation takes the form:

$$\boxed{\frac{d}{dt}\int_{Vc}\rho\,dv + \int_{Sc}\rho(\vec{V}.\vec{n})ds = 0}$$

We have $\int_{Sc}\rho(\vec{V}.\vec{n})ds = \sum q_m$ sum of mass flows.

If the control volume is fixed, $\vec{V}_{s} = 0$ and $\vec{V}_{rel} = \vec{V}$ (the flow velocity) and the flow is **steady-state, then:

$$\frac{d}{dt}\int_{Vc}\rho\,dv + \int_{Sc}\rho(\vec{V}.\vec{n})ds = 0$$

$$\int_{Sc}\rho(\vec{V}.\vec{n})ds = 0$$

$$\frac{d}{dt}\int_{Vc}\rho\, dv = 0$$

If the fluid is **incompressible ρ=cte, then we have, **even in unsteady conditions,**

$$\int_{Sc}\rho(\vec{V}.\vec{n})ds = 0 \qquad , \frac{d}{dt}\int_{Vc}\rho\, dv = 0$$

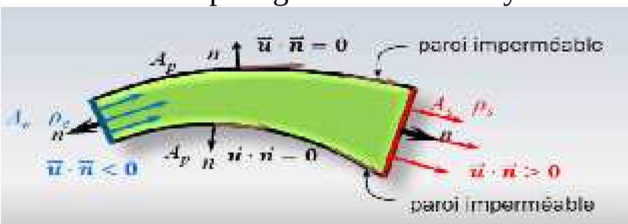
In the case of uniform inputs and outputs (1D), the above equations become :

$$\int_{Sc}\rho(\vec{V}.\vec{n})ds = 0 \Rightarrow \sum_{sortieS}\rho_i V_i S_i = \sum_{entrées}\rho_j V_j S_j \Rightarrow \sum_{sortie} qm_i = \sum_{entrée} qm_j \text{ (Mass flow rates)}$$

$$\int_{Sc}(\vec{V}.\vec{n})ds = 0 \Rightarrow \sum_{sortieS} V_i S_i = \sum_{entrées} V_j S_j \Rightarrow \sum_{sortie} qv_i = \sum_{entrée} qv_j \text{ (Volumetric flow rates)}$$

In practice, we often find applications with a single inlet and outlet, such as a pipe carrying water, or a ventilation passage in a building. These types of problems are modeled using the notion of **a current tube.**

This is a conceptually fictitious pipe (sometimes corresponding to **a physical tube**) with an inlet cross-section Ae, an outlet cross-section As , both flat, and side walls Ap tangent to the velocity vector.



Given that at the side walls $\vec{V}.\vec{n}$ =0 the surface integral only needs to consider the inlet and outlet, i.e. :

$$\int_{Sc}\rho(\vec{V}.\vec{n})dA = \int_{Se}\rho(\vec{V}.\vec{n})dA + \int_{Ss}\rho(\vec{V}.\vec{n})dA + \underbrace{\int_{Sc}\rho(\vec{V}.\vec{n})dA}_{=0}$$

In incompressible conditions (ρ=cste**)** and if the velocities u_e and u_s are considered uniform, then..:

$$\int_{Se}(\vec{V}.\vec{n})dA = -u_e A_e = Qv_e$$

$$\int_{Ss}(\vec{V}.\vec{n})dA = u_s A_s = Qv_s$$

$$\Rightarrow u_s A_s = u_e A_e \Rightarrow Qv_e = Qv_s \text{ (Constant volume flows)}$$

When $\rho \neq$ **cste,** we have $\rho_e u_e A_e = \rho_s u_s A_s$

II.7 Conservation of momentum

II.7.1 Application of the TTR for momentum:

$$\boxed{\left.\frac{dB}{dt}\right|_{système} = \frac{d}{dt}\int_{Vc}\rho b\, dv + \int_{Sc}\rho b(\vec{V}_{rel}.\vec{n})ds} \quad \textbf{with } \vec{V}_{rel} = \vec{V}$$

For momentum, $B = m\vec{V}$ and therefore $b = \frac{dB}{dm} = \vec{V}$ the TTR gives :

$$\boxed{\left.\frac{d(m\vec{V})}{dt}\right|_{\text{système}} = \frac{d}{dt}\int_{Vc} \rho\vec{V}dv + \int_{Sc} \rho\vec{V}(\vec{V}.\vec{n})ds}$$

$$\left.\frac{dB}{dt}\right|_{\text{système}} = \sum \vec{F}_{syst}$$

So $$\sum \vec{F}_{syst} = \frac{d}{dt}\int_{Vc} \rho\vec{V}dv + \int_{Sc} \rho\vec{V}(\vec{V}.\vec{n})ds$$

(since at an instant t, the material volume (in motion) coincides with the control volume, (fixed) we have the expression):

$$\sum \vec{F}_{VC} = \frac{d}{dt}\int_{Vc} \rho\vec{V}dv + \int_{Sc} \rho\vec{V}(\vec{V}.\vec{n})ds$$

Forces exerted on the VC

Variation in the qtty of M in the VC

Flow of M qte through the SC

Note that forces on the control volume are sources (+) or sinks (-) of momentum. A source (force experienced by the fluid) corresponds to an increase in its momentum. A sink (force exerted by the fluid) corresponds to a decrease in its momentum**.**

The momentum equation is a vector equation, so it can be written for the 3 velocity components *u, v* and *w*, or in index notation for *Vi* with *i=1,2,3.*

$$\sum \vec{F}_{VC} = \frac{d}{dt}\int_{Vc} \rho\vec{V}dv + \int_{Sc} \rho\vec{V}(\vec{V}_{rel}.\vec{n})ds$$

$$\sum \vec{F}_{syst} = \frac{d}{dt}\int_{Vc} \rho\vec{V}_i dv + \int_{Sc} \rho\vec{V}_i(\vec{V}_{rel}.\vec{n})ds$$

With V_i are the components u,v and w

F_i are F_x , F_y , F_z

If the control volume does not deform, $\vec{V}_{rel} = \vec{V}$ and if the flow is permanent, we have:

$$\sum \vec{F}_{VC} = \frac{d}{dt}\int_{Vc} \rho\vec{V}dv + \int_{Sc} \rho\vec{V}(\vec{V}_{rel}.\vec{n})ds$$

If, moreover, the inputs and outputs have **uniform speeds**, the integral over the *SC* is replaced by the balance of the incoming and outgoing flows, then,

$$\int_{Sc} \rho\vec{V}(\vec{V}.\vec{n})ds = \vec{V}\int_{SC} \underbrace{\rho\vec{V}.\vec{n}ds}_{qm} \text{ So } \sum \vec{F}_{VC} = \sum_{\text{Sortie}} qm_i\vec{V}_i - \sum_{\text{Entrés}} qm_j\vec{V}_j$$

Note: The summations over *i* and *j* correspond to the input/output numbers

In the case of a single input and output we have:

$$\boxed{\sum \vec{F}_{VC} = q_m(\vec{V}_{sortie} - \vec{V}_{entrée})}$$

We can use the current tube as an example.

$$\sum \vec{F}_{VC} = q_m(\vec{V}_{sortie} - \vec{V}_{entrée})$$

$$\sum \vec{F}_{VC} = (\rho_2 V_2 A_2)\vec{V}_2 - (\rho_1 V_1 A_1)\vec{V}_1$$

$$\sum \vec{F}_{VC} = \vec{P} + \vec{F}p_1 + \vec{F}p_2 + \vec{F}pp + \vec{F}rp$$