Chapter III

Dimensional analysis and similarities

III.1 <u>Introduction:</u>

Dimensional analysis is a practical method for verifying the homogeneity of a physical formula through its dimensional equations, i.e. the decomposition of the physical quantities it involves into a product of basic quantities: length, duration, mass, electrical intensity, etc., which are irreducible to each other**.**

Dimensional analysis enables :

- Determine the unit of a quantity
- Check the homogeneity of a formula
- Predict the form of a physical law in order to find the solution to certain problems without having to solve an equation: for many of the physical phenomena studied, we can express a magnitude characteristic of the phenomenon and deduce an order of magnitude.

Dimensional analysis can be applied in almost every field of engineering. What's more, it's a highly useful additional tool in modern fluid mechanics. It is based on the principle of dimensional homogeneity and uses the dimensions of the relevant variables affecting the phenomenon in question.

III.2. <u>Dimensions:</u>

The various physical quantities used in fluid mechanics can be expressed in terms of fundamental or primary quantities.

In the International System, the primary or fundamental physical quantities are mass, length, time and sometimes temperature (compressible flows) and are designated respectively by the letters **M,L,T,θ.** Quantities that are expressed as a function of fundamental quantities are called secondary or derived quantities (velocity, area acceleration....). The expression of a derived quantity as a function of the fundamental quantity is called the Dimension of the physical quantity.

A quantity can be expressed dimensionally as M,L,T or F,L,T.

For example:

Flow =speed x area = $\frac{L}{T}.L^2 \; Z \; \frac{L^3}{T} \; Z \; L^3.T^{J1}$

The kinematic viscosity ν=⇑ /ρ is $\varnothing Z \Uparrow \frac{du}{dy}$ and $\Uparrow \; Z \; \frac{\varnothing}{\frac{du}{dy}} \; Z \; \frac{\text{contra int e}}{\frac{L}{T} \text{x} \frac{1}{L}} \; Z \; \frac{\text{force / aire}}{\frac{1}{T}}$

$$= \frac{\text{massexaccélération}}{\text{airex}\frac{1}{\text{T}}} \text{ Z } \frac{\text{Mx}\frac{\text{L}}{\text{T}^2}}{\text{L}^2\text{x}\frac{1}{\text{T}}} \text{ Z } \frac{\text{ML}}{\text{L}^2\text{T}^2\text{x}\frac{1}{\text{T}}} \text{ Z } \frac{\text{M}}{\text{LT}} \text{ Z } \text{ML}^{\text{J}1}\text{T}^{\text{J}1}$$

$$\partial \text{ Z } \frac{\text{masse}}{\text{volume}} \text{ Z } \frac{\text{M}}{\text{L}^3} \text{ Z } \text{ML}^{\text{J}3}$$

Thus the kinematic viscosity $\Rightarrow \text{Z } \frac{\Uparrow}{\partial} \text{ Z } \frac{\text{ML}^{\text{J}1}\text{T}^{\text{J}1}}{\text{ML}^{\text{J}3}} \text{ Z } \text{ML}^{\text{J}2}\text{T}^{\text{J}2}$

III.3. Principle of dimensional homogeneity

An equation is considered to be dimensionally homogeneous if the form of the equation does not depend on the units of measurement, or if the two terms of the equation have the same dimensions.

III.4 Dimensional analysis method

Dimensions can be used to determine whether a literal expression is homogeneous or not. This allows us to search for possible errors. But dimensional analysis can also be used to find or guess at physical laws when theoretical resolution is too complex.

When the system under study is **too complex** to allow complete resolution of the fundamental equations, or when its behavior is chaotic, dimensional analysis provides simple access to relationships between the various quantities characterizing the system.

Grouping these different quantities into **dimensionless numbers** will also enable us to establish **similarities** between the behavior of **similar** but different **systems** (prototype/model).

The application of dimensional analysis to a practical problem is based on the assumption that certain variables affecting the phenomenon are independent. The number of variables characterizing the problem is equal to the number of independent variables plus one. One is the number of dependent variables.

Dimensional analysis is used to obtain a functional relationship between dependent and independent variables.

The first step in dimensional analysis is to determine the variables involved in the problem. Naming these variables requires a good understanding of the phenomenon. The second step is to form adimensional groups of these variables.

The Vachy-Buckingham method -π(π Buckingham theorem)is the most widely used method in dimensional analysis.

Let's take the example of determining regular head losses in a cylindrical pipe:

The various quantities involved are :

$\frac{\text{a}P_t}{L}$ Pressure drop per unit length,

D Pipe diameter,

ε Pipe roughness,
v Average flow velocity (or flow rate),
μ Fluid viscosity,
ρ The density of the fluid.
Consequently, there is a relationship between these different quantities:

$$\frac{aP_t}{L} \text{ Z } f(D, £, v, «, °)$$

The function *f* may be difficult to find, so dimensional analysis will enable us to establish a simpler relationship between a smaller number of dimensionless quantities.
A systematic method will find 3 dimensionless numbers:

$$\Leftrightarrow_1 \text{ Z } \frac{\zeta p_t}{L}\frac{D}{\partial V^2} \text{ Z } \Leftarrow \qquad \Leftrightarrow_2 \text{ Z } \frac{\partial VD}{\Uparrow} \text{ Z Re} \qquad \Leftrightarrow_3 \text{ Z } \frac{\aleph}{D} \text{ Z } \aleph_1$$

In this way, we can establish : $\Leftrightarrow_1 \text{ Z } \theta\,(\Leftrightarrow_2, \Leftrightarrow_3)$

$$\hat{G}\ \frac{\zeta p_t}{L}\frac{D}{\partial V^2} \text{ Z } \theta \left(\frac{\partial VD}{\Uparrow}, \frac{\aleph}{D}\right) \hat{G}\ \frac{\zeta p_t}{L} \text{ Z } \frac{\partial V^2}{D}\theta\,(\text{Re}, \aleph_1)$$

Dimensional analysis shows that regular pressure drop is a function of **Reynolds** number and relative pipe roughness alone.

III.5. ***Vachy-Buckingham*'s π theorem**

The π-Buckingham method expresses the resulting equation in terms of adimensional groups (π-terms). According to this theorem, if a phenomenon drives p variables: a a $a_{1,2,3}$,.....,.a_p such that one variable a_1 depends on the other independent variables a $a_{2,3}$,.....,.a_n , the general functional relationship between dependent and independent variables can be expressed as follows:

a_1 =f(a $a_{2,3}$,...................a_p) (III.1)

Expression (III.1) can be written mathematically as :

$\leftarrow$ (a $a_{2,3}$,.........a_p)=0 (III.2)

That is, if an equation with *p* variables is homogeneous, it can be reduced to a relationship between (*p-q*) dimensionless independent products, where *q* is the minimum number of dimensions required to describe the *p* variables, and we write:

$f(\pi_1 , \pi_2 , \pi_3 ,\ldots\ldots\ldots \pi_{p-q})=0$ (III.3)

In problems where all fundamental dimensions are considered, we recommend selecting repeated variables using the following guidelines:

- Select the first variable repeated from those describing the flow geometry.
- Select the second variable repeated from those representing fluid properties.
- Select the third variable repeated from those characterizing fluid movement.

To illustrate this statement, let's return to the previous example:
We had p=6 variables (requiring a minimum of *q=3* dimensions (M,L,T).

$$\left[\frac{\Delta p_t}{L}\right] = ML^{-2}T^{-2}$$

$$[D] = L$$

$$[\varepsilon] = L$$

$$[V] = LT^{-1}$$

$$[\mu] = ML^{-1}T^{-1}$$

$$[\rho] = ML^{-3}$$

Consequently, the equation linking the 6 variables can be reduced to an equation linking p-q = 3 dimensionless products:

$$\pi_1 = \frac{\Delta p_t}{L}\frac{D}{\rho V^2} = \lambda \qquad \pi_2 = \frac{\rho V D}{\mu} = Re \qquad \pi_3 = \frac{\varepsilon}{D} = \varepsilon_r$$

Buckingham's π theorem, therefore allows the passage :

$$\frac{\Delta p_t}{L} = f(D, \varepsilon, V, \mu, \rho) \qquad \pi_1 = \varphi(\pi_2, \pi_3)$$

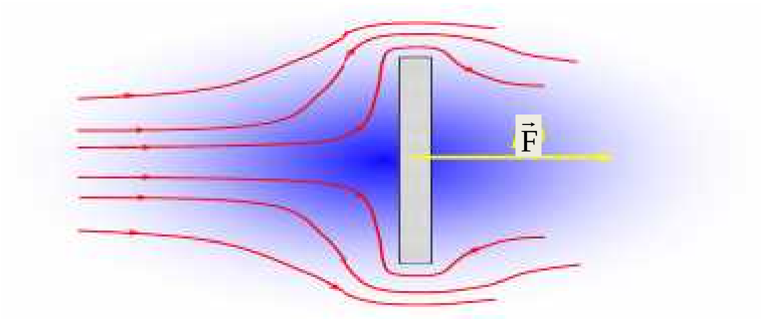
To apply this theorem, we need to use a systematic method:

- *List the variables in the problem p*
- *Write the equation in dimensions for each of the p variables*
- *Determine **q**, and thus **p-q** the number of dimensionless products characterizing the problem.*
- *Among the **p** variables, choose a number **q** that are dimensionally independent q primary variables*
- *Form the **p-q** products* π *by combining the **p-q** non-primary variables with the **q** primary ones to obtain dimensionless quantities.*
- *Formulate the relationship between the **p-q** products* π *found.*

We'll apply the method to the example of flow around a vertical plate, to write the drag force exerted by the flow on the plate in dimensionless form.

The force $\vec{F}$ exerted by a flow on an object in the direction parallel to the flow is called the drag force. Let's look at a rectangular flat plate.

The problem variables are :



F,h,L,μ ,ρ,v ⇒ ***p = 6***

F: drag force
h: plate height
L: plate width
v mean flow velocity
μ fluid viscosity ρ :fluid density

1. Variables F,h,L,V,μ ,ρ **p = 6 M,L**,T q=3
2. Dimensional equations :

$$\left.\begin{aligned}[F] &= MLT^{-2}\\ [\mu] &= ML^{-1}T^{-1}\\ [V] &= LT^{-1}\\ [h] &= L\\ [L] &= L\\ [\rho] &= ML^{-3}\end{aligned}\right\}$$

3. Number of products π dimensionless: (*p*- *q) = 6* - *3 = 3*

4. Choice of *q= 3* dimensionally independent primary *variables:*

(For example *h,*ρ and *v)*

5. Formation of the 3 π products: by combining primary and non-primary variables.

$$\pi_2 = L\, h^{a_2}\rho^{b_2}v^{c_2} \qquad \pi_3 = \mu\, h^{a_3}\rho^{b_3}v^{c_3}$$

$$\pi_1 = F\, h^{a_1}\rho^{b_1}v^{c_1}$$

6/Formulate the relationship between the **3** products π found:

$$F = f(h, L, v, \mu, \rho) \qquad \Rightarrow \quad \pi_1 = \varphi(\pi_2, \pi_3)$$

With $\pi_3 = \dfrac{\mu}{\rho v h}$ $\pi_1 = \dfrac{F}{\rho v^2 h^2}$, $\pi_2 = \dfrac{L}{h}$

Or : $F = \rho v^2 h^2\ \varphi(L/h, 1/Re)$

ý $\downarrow$

Form factor

Nature of flow

III.5.1 Illustrating the benefits of the method :

If F_1 is the drag force measured on a plate of dimensions L_1 x h_1 when subjected to a flow of velocity v_1 , then :

$$\frac{F_1}{\rho v_1^2 h_1^2} = \theta\,(L_1/h_1\,, 1/Re_1) \text{ Where } Re_1 = \frac{\rho v_1 h_1}{\mu}$$

Dimensional analysis using *Buckingham*'s theorem shows that for a plate with dimensions L_2 x h_2 such that :

$$L_2/h_2 = L_1/h_1 \quad \text{Shape similarity}$$

$$\text{If } v_2 = \frac{h_1}{h_2} v_1 \;\Rightarrow\; v_1 h_1 = v_2 h_2 \;\Rightarrow\; Re_1 = Re_2 \text{ hydrodynamic similarity}$$

$\downarrow$

Scale factor

And so $\theta\,(L_1/h_1\,, 1/Re_1) = \theta\,(L_2/h_2\,, 1/Re_2)$

$$\Rightarrow\ \frac{F_1}{\rho v_1^2 h_1^2} = \frac{F_2}{\rho v_2^2 h_2^2} \ \Rightarrow\ F_2 = \frac{v_2^2 h_2^2}{v_1^2 h_1^2} F_1 \qquad \Rightarrow\ F_2 = F_1$$

III.6 Common dimensionless coefficients

There are a number of dimensionless quantities which can characterize the nature of a flow:

- ***Reynolds* number =** $Re = \dfrac{\rho VL}{\mu}$ $\dfrac{\textbf{forces d'inertie}}{\textbf{forces de viscosité}}$ (III.4)

  General importance for all types of flow

- ***Froude* number =** $Fr = \dfrac{V}{\sqrt{gL}}$ $\dfrac{\textbf{forces d'inertie}}{\textbf{forces de gravité}}$ (III.5)

Importance for free-surface flows.

- ***Euler*** **number =** $Eu = \dfrac{\Delta p}{\rho V^2} = \dfrac{\textbf{forces de pression}}{\textbf{forces de d'inertie}}$ (III.6)

Important if there are large pressure differences within the flow.

- ***Mach*** **number** $Ma = \dfrac{V}{c} = \dfrac{\textbf{forces d'inertie}}{\textbf{forces de compressibilité}}$ (III.7)

Importance for compressible fluid flows.

With $c = 1/\sqrt{\rho \Re}$ the speed of sound.

- ***Strouhal*** **number =** $St = \dfrac{\omega L}{V} = \dfrac{\textbf{forces d'inertie locales}}{\textbf{forces d'inertie convectives}}$ (III.8)

Importance for non-stationary flows

III.7. Similarity in differential equations

To carry out a complete analysis of a flow, we first need to make the appropriate simplifying assumptions. Evaluating the various dimensionless flow coefficients (*Reynolds*, *Froude*, *etc.*) will simplify the equations to be solved.
Let's consider the conservation of momentum equations (the Navier-Stockes equations) for an incompressible flow along x, y and z, written in the form:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\rho g$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\rho g$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\rho g$$

Let's consider the (z) component of these equations and write it in dimensionless form:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\rho g$$

Let's introduce some dimensionless variables:

$$\begin{cases} u^* = u/V \\ v^* = v/V \\ w^* = w/V \end{cases} \quad \begin{cases} x^* = x/L \\ y^* = y/L \\ z^* = z/L \end{cases} \quad p^* = p/p_0 \ \text{ and } \ t^* = t/\tau$$

Where *L*, *V*, *p* ,$_0$² are characteristic quantities of the system under study.

We have :

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{1}{L}\dfrac{\partial}{\partial x^*} & \dfrac{\partial^2}{\partial x^2} = \dfrac{1}{L^2}\dfrac{\partial^2}{\partial x^{*2}} \\ \dfrac{\partial}{\partial y} = \dfrac{1}{L}\dfrac{\partial}{\partial y^*} & \dfrac{\partial^2}{\partial y^2} = \dfrac{1}{L^2}\dfrac{\partial^2}{\partial y^{*2}} \quad \dfrac{\partial}{\partial t} = \dfrac{1}{\tau}\dfrac{\partial}{\partial t^*} \\ \dfrac{\partial}{\partial z} = \dfrac{1}{L}\dfrac{\partial}{\partial z^*} & \dfrac{\partial^2}{\partial z^2} = \dfrac{1}{L^2}\dfrac{\partial^2}{\partial z^{*2}} \end{cases}$$

By replacing in our equation we obtain :

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - \rho g$$

$$\frac{\rho V}{\tau}\frac{\partial w^*}{\partial t^*} + \frac{\rho V^2}{L}\left(u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}\right)$$
$$= -\frac{p_0}{L}\frac{\partial p^*}{\partial z^*} + \frac{\mu V}{L^2}\left(\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}}\right) - \rho g$$

$$\frac{\rho V}{\tau}\frac{\partial w^*}{\partial t^*} + \frac{\rho V^2}{L}\left(u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}\right)$$
$$= -\frac{p_0}{L}\frac{\partial p^*}{\partial z^*} + \frac{\mu V}{L^2}\left(\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}}\right) - \rho g$$

Divide the whole expression by $\dfrac{\rho V^2}{L}$:

$$\frac{L}{V\tau}\frac{\partial w^*}{\partial t^*} + u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}$$
$$= -\frac{p_0}{\rho V^2}\frac{\partial p^*}{\partial z^*} + \frac{\mu}{\rho V L}\left(\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}}\right) - \frac{gL}{V^2}$$

$$St = \frac{\omega L}{V} = \frac{L}{V\tau} \quad Eu = \frac{\Delta p}{\rho V^2} = \frac{p_0}{\rho V^2} \quad Re = \frac{\rho V L}{\mu} \quad Fr = \frac{V}{\sqrt{gL}}$$

We can then write :

$$St\frac{\partial w^*}{\partial t^*} + u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}$$
$$= -Eu\frac{\partial p^*}{\partial z^*} + \frac{1}{Re}\left(\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}}\right) - \frac{1}{Fr^2}$$

This can be interpreted as follows:

- If ***St*** is very small: the instantaneous derivative can be neglected and the flow can be considered stationary.
- If ***Eu*** is very low, the pressure gradient can be neglected.
- If ***Re*** is very large: we can neglect the fluid's viscosity and treat it as a perfect fluid.
- If ***Fr*** is very large: the effects of gravity can be neglected.

III.8. Similarity and model tests

To find out about the performance of mechanical or hydraulic structures or machines (pumps, turbines, ...) before they are built or manufactured, the study is carried out on a model, which is a representation on a different scale of the system or structure (prototype) to be tested.

- **The** small-scale **model** reproducing the current structure
- **The prototype** is the structure or machine



Virtual model Laboratory model Prototype

The study of fluid mechanics and hydraulics problems leads to:

- Geometric similarity
- Kinematic similarity
- Dynamic similarity

III.8.1 Geometric similarity

For there to be a geometric similarity between a model and a prototype, the length ratios must be the same, and the angles between the dimensions must also be the same.

L_m : model length

H_m : model height

D_m : model diameter

A_m : the area of the model

v_m : model volume

And let L_p ,H_p ,D_p ,A_p and v_p be the corresponding prototype values.

For a geometric similarity we have :

$$\frac{L_m}{L_p} = \frac{H_m}{H_p} = \frac{D_m}{D_p} = L_r \tag{III.9}$$

The$_r$ is called the **scale factor**

$$\frac{A_m}{A_p} = A_r A_r \text{ is the ratio of areas} \tag{III.10}$$

$$\frac{v_m}{v_p} = v_r v_r \text{ is the volume ratio} \tag{III.11}$$

III.8.2 Kinematic similarity

Kinematic similarity is similarity of motion.

If, at the points corresponding to the model and the prototype, the velocity and acceleration ratios are the same, as well as the velocity in the same directions, the two flows are said to be kinematically similar.

$(V)_{1m}$: fluid velocity at point 1 of the model

$(V)_{2m}$: fluid velocity at point 2 of the model

$(a)_{1m}$: fluid acceleration at model point 1

$(a)_{2m}$: fluid acceleration at model point 2

And $(V)_{1p}$,$(V)_{2p}$,$(a)_{1p}$,$(a)_{2p}$, the corresponding velocities and accelerations at points in the prototype fluid.

For a kinematic similarity we have :

$$\frac{(V_1)_m}{(V_1)_p} = \frac{(V_2)_m}{(V_2)_p} = V_r \tag{III.12}$$

V_r velocity ratio

$$\frac{(a_1)_m}{(a_1)_p} = \frac{(a_2)_m}{(a_2)_p} = a_r \tag{III.13}$$

a_r the acceleration ratio

The direction of velocity in the model and in the prototype must be the same.

III.8.3 Dynamic similarity

Dynamic similarity is similarity of forces. The forces in the model and the prototype are similar.

If at the corresponding points, identical types of force are parallel and give the same ratio.

$(F)_{im}$: the inertial force at the model point

$(F)_{vm}$: viscous force at the model point

$(F)_{gm}$: the force of gravity at the model point

And (F) ,$_{ip}$ $(F)_{vp}$, $(F)_{gp}$ are the forces corresponding to the prototype.

For a dynamic similarity we have :

$$\frac{(F_i)_m}{(F_i)_p} = \frac{(F_v)_m}{(F_v)_p} = \frac{(F_g)_m}{(F_g)_p} = F_r \qquad \text{(III.14)}$$

F_r the balance of power

The directions of the forces in the model and in the prototype must be the same.

To ensure dynamic similarity between model and prototype, the dimensionless numbers of the model and prototype must be the same.

This condition cannot be satisfied for all dimensionless numbers, so models are designated on the basis of the forces that dominate them. This flow situation is called **the law of similitude.**

- **Reynolds Model Law**

In a flow situation where, in addition to inertial forces, viscous forces predominate. The flow similarity between the model and the prototype can be established if the Reynolds numbers are the same for both systems.

$(R_e)_{model} = (R_e)_{prototype}$ (III.15)

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p} \Rightarrow \frac{\rho_m}{\rho_p}.\frac{V_m}{v_p}.\frac{L_m}{L_p}\frac{1}{\frac{\mu_m}{\mu_p}} = 1$$

We have $$\frac{\rho_r V_r L_r}{\mu_r} = 1 \qquad \text{(III.16)}$$

And $\left\{\rho_r = \frac{\rho_p}{\rho_m}, V_r = \frac{V_p}{V_m}, L_r = \frac{L_p}{L_m}\right\}$

Where the different index quantities r represent the scaling ratios.

Similarly, we have :

Time scale $T_r = \frac{L_r}{V_r}$ (III.17)

The scale of acceleration $a_r = \frac{V_r}{T_r}$

The force scale F_r = (mass X acceleration)= $m_r .a = \rho\ A_{rrrr}\ V.a_{r=}\ \rho\ L\ V_{rr}^{22}{}_r$. (III.18)

The flow rate scale $q_r = (\rho AV) = \rho\ A_{rrr} = V\ \rho\ L\ V_{rr}^{2}{}_r$ (III.19)