حل السلسلة رقم 02

مقياس : التحويلات التكاملية Transformations intégrales

Exercice 03 :

$a > 0$

1) $f_a(x) = e^{-ax} \cdot \chi_{]0,+\infty[}(x) = \begin{cases} e^{-ax}, & x > 0 \\ 0, & x \leq 0 \end{cases}$, $\varphi_k(x) = \dfrac{x^k}{k!} f_a(x), k \in \mathbb{N}$.

on a : $\mathcal{F}(f_a(x))(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} \cdot f_a(x)\, dx = \int_0^{+\infty} e^{-2\pi i x \xi} \cdot e^{-ax}\, dx$

$= \dfrac{-1}{a+2\pi i \xi} \left[e^{-(2\pi i \xi + a)x} \right]_0^{+\infty} = \dfrac{1}{a+2\pi i \xi}$

donc, $\boxed{\hat{f}_a(\xi) = \dfrac{1}{a+2\pi i \xi}}$

on a : $\dfrac{d^{(k)}}{d\xi^k} \hat{f}_a(\xi) = \mathcal{F}\left((-2\pi i x)^k f_a(x) \right)(\xi)$

$\Rightarrow \dfrac{1}{k!} \dfrac{d^{(k)}}{d\xi^k} \hat{f}_a(\xi) = (-2\pi i)^k \mathcal{F}\left(\dfrac{x^k}{k!} f_a(x) \right)(\xi)$ (car $\mathcal{F}$ est linéaire)

$\Rightarrow \boxed{\mathcal{F}(\varphi_k(x))(\xi) = \dfrac{1}{(-2\pi i)^k k!} \times \dfrac{d^{(k)}}{d\xi^k} \hat{f}_a(\xi) \text{ --- } (*)}$

on utilisons la dérivée successive, on obtient

$\dfrac{d^{(1)}}{d\xi} \hat{f}_a(\xi) = \dfrac{-2\pi i}{(a+2\pi i \xi)^2} = \dfrac{(-2\pi i)^1 \times 1!}{(a+2\pi i \xi)^2}$

$\dfrac{d^{(2)}}{d\xi^2} \hat{f}_a(\xi) = \dfrac{(-2\pi i)^2\, 2}{(a+2\pi i \xi)^3} = \dfrac{(-2\pi i)^2 \times 2!}{(a+2\pi i \xi)^3}$

$$\frac{d^{(3)}}{d\xi^3} = \frac{(-2\pi i)\cdot 6}{(a+2\pi i \xi)^4} = \frac{(-2\pi i)\cdot 3!}{(a+2\pi i \xi)^4}$$

donc, on peut démontrer par la récurrence que

$$\frac{d^{(k)}}{d\xi^k} \hat{f}_a(\xi) = \frac{(-2\pi i)^k \cdot k!}{(a+2\pi i \xi)^{k+1}}$$

Alors, $(*) \iff$

$$\boxed{\mathcal{F}(\varphi_k(x))(\xi) = \frac{1}{(a+2\pi i \xi)^{k+1}}} = \hat{\varphi}_k(\xi)$$

$$\boxed{\mathcal{F}(\check{f}_a(x))(\xi) = \hat{f}_a(-\xi) = \frac{1}{a-2\pi i \xi}}$$

et

$$\boxed{\mathcal{F}(\check{\varphi}_k(x))(\xi) = \hat{\varphi}_k(-\xi) = \frac{1}{(a-2\pi i \xi)^{k+1}}}$$

2) $\mathcal{F}(g_a(x))(\xi) = \mathcal{F}(f_a(x))(\xi) + \mathcal{F}(f_a(-x))(\xi)$, car $\mathcal{F}$ est linéaire

$$= \hat{f}_a(\xi) + \hat{f}_a(-\xi)$$

$$= \frac{1}{a+2\pi i \xi} + \frac{1}{a-2\pi i \xi} = \frac{2a}{a^2+4\pi^2\xi^2} = \hat{g}_a(\xi)$$

* pour déduire la valeur de l'intégrale, on utilisons la transformée de Fourier inverse, alors on a

$$g_a(x) = \mathcal{F}^{-1}(\hat{g}_a(\xi))(x) = \int_{\mathbb{R}} e^{2\pi i x\cdot\xi} \cdot \hat{g}_a(\xi)\, d\xi$$

$$\Rightarrow \int_{\mathbb{R}} \left(\cos(2\pi x\xi) + i\sin(2\pi x\xi)\right) \cdot \hat{g}_a(\xi)\, d\xi = g_a(x) = e^{-a|x|}$$

car, $g_a(x) = f_a(x) + f_a(-x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases} + \begin{cases} e^{ax}, & x \leq 0 \\ 0, & x > 0 \end{cases} = e^{-a|x|}$

$$\Rightarrow 2\int_0^{+\infty} \cos(2\pi x\xi)\, \hat{g}_a(\xi)\, d\xi = e^{-a|x|}, \text{ car}$$

la fonction $\cos(2\pi x\xi)\, \hat{g}_a(\xi)$ est paire et $\sin(2\pi x\xi)\, \hat{g}_a(\xi)$ est impaire

$$\Rightarrow 4a \int_0^{+\infty} \frac{\cos(2\pi x\xi)}{a^2 + 4\pi^2\xi^2}\, d\xi = e^{-a|x|}$$

choisissons, $a = 2\pi$, $w = 2\pi x$, on obtient

$$8\pi \int_0^{+\infty} \frac{\cos(w\xi)}{4\pi^2 + 4\pi^2\xi^2}\, d\xi = e^{-2\pi|x|} = e^{-|w|}$$

$$\Rightarrow \boxed{\int_0^{+\infty} \frac{\cos(w\xi)}{1+\xi^2}\, d\xi = \frac{\pi}{2} \cdot e^{-|w|}}$$

Exercice 04 :

1) supposons que $\exists g \in L^1(\mathbb{R}) : g * f = f, \ \forall f \in L^1(\mathbb{R})$

$$\Rightarrow \mathcal{F}(g * f)(\xi) = \mathcal{F}(f)(\xi)$$

$$\Rightarrow \hat{g}(\xi) \cdot \hat{f}(\xi) = \hat{f}(\xi) \Rightarrow \hat{f}(\xi)\left[\hat{g}(\xi) - 1\right] = 0$$

$$\Rightarrow \begin{cases} \hat{f}(\xi) = 0 \\ \vee \\ \hat{g}(\xi) = 1 \end{cases} \Rightarrow \lim_{\xi \to \pm\infty} \hat{g}(\xi) = 1, \text{ contradiction}$$

(3)

Car, si $g \in L^1(\mathbb{R})$, on a $\lim\limits_{\xi \to \pm\infty} \hat{g}(\xi) = 0$.

donc, $\nexists\, g \in L^1(\mathbb{R})$: $g * f = f$, $\forall f \in L^1(\mathbb{R})$.

2) on a $f * f = f \Rightarrow \mathcal{F}(f*f)(\xi) = \mathcal{F}(f)(\xi)$

$\Rightarrow \hat{f}(\xi) \cdot \hat{f}(\xi) = \hat{f}(\xi) \Rightarrow \hat{f}(\xi)\left[\hat{f}(\xi) - 1\right] = 0$

$\Rightarrow \begin{cases} \hat{f}(\xi) = 0 \\ \vee \\ \hat{f}(\xi) = 1 \end{cases}$ impossible, car $\lim\limits_{\xi \to \pm\infty} \hat{f}(\xi) = 0$

donc, $\hat{f}(\xi) = 0 \Rightarrow \boxed{f = 0 \ \text{p.p.}}$

EXO 5 :

$f(x) = e^{-\pi x^2}$

1) on a : $f'(x) + 2\pi x f(x) = -2\pi x e^{-\pi x^2} + 2\pi x e^{-\pi x^2} = 0$

2) on a : $\mathcal{F}(f'(x))(\xi) + 2\pi \mathcal{F}(x f(x))(\xi) = 0$, car $\mathcal{F}$ est linéaire

$\Rightarrow (2\pi i \xi)\hat{f}(\xi) + 2\pi\left(\frac{-1}{2\pi i}\frac{d}{d\xi}\hat{f}(\xi)\right) = 0$

$\Rightarrow (2\pi i \xi)\hat{f}(\xi) - \frac{1}{i}\frac{d}{d\xi}\hat{f}(\xi) = 0$

$\Rightarrow \boxed{\frac{d}{d\xi}\hat{f}(\xi) + 2\pi\xi\hat{f}(\xi) = 0} \cdots \hat{(1)}$

3) on a $\frac{d}{d\xi}\hat{f}(\xi) + 2\pi\xi\hat{f}(\xi) = 0 \Rightarrow \frac{d}{d\xi}\hat{f}(\xi) = -2\pi\xi\hat{f}(\xi)$

$$\Rightarrow \frac{d\hat{f}(\xi)}{\hat{f}(\xi)} = -2\pi\xi\, d\xi \Rightarrow \int \frac{d\hat{f}(\xi)}{\hat{f}(\xi)} = \int -2\pi\xi\, d\xi$$

$$\Rightarrow \ln\left(\frac{\hat{f}(\xi)}{c}\right) = -\pi\xi^2 \Rightarrow \frac{\hat{f}(\xi)}{c} = e^{-\pi\xi^2}$$

$$\Rightarrow \boxed{\hat{f}(\xi) = c\,e^{-\pi\xi^2}}$$

on a $\boxed{\hat{f}(0) = c}$ et $\hat{f}(0) = \int_{\mathbb{R}} e^{-2\pi i x(0)} \cdot f(x)\,dx$

$$\Rightarrow c = \int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} e^{-\pi x^2}\,dx = 1$$

donc, $\boxed{\hat{f}(\xi) = e^{-\pi\xi^2}}$

Exercice 06 : $\varphi_{fg}(x) = \int_{\mathbb{R}} f(y) g(x+y)\,dy$

on a :

$$\varphi_{fg}(x) = \int_{\mathbb{R}} f(y)\left(\int_{\mathbb{R}} e^{2\pi i (x+y)\cdot\xi} \cdot \hat{g}(\xi)\,d\xi\right)dy$$ → d'après la définition de $\mathcal{F}^{-1}$

$$= \int_{\mathbb{R}} \hat{g}(\xi)\left(\int_{\mathbb{R}} f(y)\, e^{2\pi i y\cdot\xi}\,dy\right) e^{2\pi i x\cdot\xi}\,d\xi$$ → D'après Fubini

$$= \int_{\mathbb{R}} \hat{g}(\xi)\cdot \overline{\hat{f}(\xi)}\, e^{2\pi i x\cdot\xi}\,d\xi$$, car f est réelle

2) pour $x = 0$, on a

$$\varphi_{fg}(0) = \int_{\mathbb{R}} f(y) g(y)\,dy = \int_{\mathbb{R}} \overline{\hat{f}(\xi)}\cdot\hat{g}(\xi)\,d\xi$$

un cas particulier, si $f = g$ on a

$$\boxed{\int_{\mathbb{R}} |f(x)|^2 \, dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, d\xi}$$

3) on a $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \cdot \xi} f(x)\, dx = \int_{-1}^{1} e^{-2\pi i x \cdot \xi} dx = \frac{-1}{2\pi i \xi}\left[e^{-2\pi i x\xi}\right]_{-1}^{1}$

$$\Rightarrow \boxed{\hat{f}(\xi) = \frac{\sin(2\pi\xi)}{\pi\xi}}$$

donc, $\mathcal{F}\left(\frac{\sin(2\pi x)}{\pi x}\right)(\xi) = f(-\xi) = f(\xi)$, car f est paire

4) on a d'après l'identité de parseval

$$\int_{\mathbb{R}} \left|\frac{\sin(2\pi x)}{\pi x}\right|^2 dx = \int_{\mathbb{R}} |f(\xi)|^2 d\xi = \int_{-1}^{1} d\xi = 2$$

on pose $2\pi x = t \Rightarrow dt = 2\pi\, dx$, alors on a

$$\frac{4}{2\pi} \int_{\mathbb{R}} \frac{\sin^2(t)}{t^2}\, dt = 2$$

$$\Rightarrow \boxed{\int_{\mathbb{R}} \frac{\sin^2(t)}{t^2}\, dt = \pi}$$