Série 3 :

Ex 1 : on a 1 seul degré de liberté
la coordonnée généralisée $q_\alpha = \phi$

$$\begin{cases} x = a(\phi - \sin\phi) \\ y = a(1 + \cos\phi) \end{cases} \Rightarrow \begin{cases} \dot{x} = a\dot{\phi}(1-\cos\phi) \\ \dot{y} = -a\dot{\phi}\cos\phi . \end{cases}$$

$$E_c = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = ma^2\dot{\phi}^2(1-\cos\phi).$$

$$U = mgy = mga(1+\cos\phi) \text{ [on prend } U(y=0)=0 \text{]}.$$

$$\mathcal{L} = E_c - U = ma^2\dot{\phi}^2(1-\cos\phi) - mga(1+\cos\phi).$$

L'hamiltonien : $H = \sum p_\alpha \dot{q}_\alpha - \mathcal{L}$, $P = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}$ $(q_\alpha = \phi)$

$$H = p\dot{\phi} - \mathcal{L} \quad , \quad P = 2ma^2\dot{\phi}(1-\cos\phi) \Rightarrow \dot{\phi} = \frac{P}{2ma^2(1-\cos\phi)}$$

$$H(\phi, P) = P\left(\frac{P}{2ma^2(1-\cos\phi)}\right) - ma^2\left(\frac{P}{2ma^2(1-\cos\phi)}\right)^2(1-\cos\phi) + mga(1+\cos\phi)$$

$$H(\phi, P) = \frac{P^2}{4ma^2(1-\cos\phi)} + mga(1+\cos\phi)$$

les équations de Hamilton :

$$\begin{cases} \dot{q}_\alpha = \frac{\partial H}{\partial P_\alpha} \\ \dot{P}_\alpha = -\frac{\partial H}{\partial q_\alpha} \end{cases} \Rightarrow \begin{cases} \dot{\phi} = \frac{\partial H}{\partial p} = \frac{P}{2ma^2(1-\cos\phi)} & (1) \\ \dot{P} = -\frac{\partial H}{\partial \phi} = \frac{P^2 \sin\phi}{4ma^2(1-\cos\phi)^2} + mga\sin\phi & (2) \end{cases}$$

de (1) $\Rightarrow P = 2ma^2(1-\cos\phi)\dot{\phi}$

$$\dot{P} = 2ma^2\ddot{\phi}(1-\cos\phi) + 2\dot{\phi}^2 ma^2 \sin\phi \quad (3)$$

$$(1) = (3) \Rightarrow 2ma^2\ddot{\phi}(1-\cos\phi) + 2ma^2\dot{\phi}^2\sin\phi = ma^2\dot{\phi}^2\sin\phi + mga\sin\phi$$

$$\Rightarrow \boxed{2\ddot{\phi}(1-\cos\phi) + 2\dot{\phi}^2\sin\phi - \frac{g}{a}\sin\phi = 0}$$

on a : $(1-\cos\phi) = 2\sin^2\frac{\phi}{2}$ et $\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$.

$$4\sin^2\frac{\phi}{2}\ddot{\phi} + 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\dot{\phi}^2 - \frac{2g}{a}\sin\frac{\phi}{2}\cos\frac{\phi}{2} = 0$$

$$\boxed{2\sin\frac{\phi}{2}\cdot\ddot{\phi} + \cos\frac{\phi}{2}\dot{\phi}^2 - \frac{g}{a}\cos\frac{\phi}{2} = 0}$$

$$U = \cos\frac{\phi}{2} \Rightarrow \frac{dU}{dt} = -\frac{\dot{\phi}}{2}\sin\frac{\phi}{2} \Rightarrow \frac{d^2U}{dt^2} = -\frac{\dot{\phi}^2}{4}\cos\frac{\phi}{2} - \frac{\ddot{\phi}}{2}\sin\frac{\phi}{2}.$$

$$\Rightarrow \quad \sin\frac{\phi}{2}\ddot{\phi} + \dot{\phi}^2\cos\frac{\phi}{2} = -4\frac{d^2U}{dt^2}$$

$$\Rightarrow \quad -4\frac{d^2U}{dt^2} - \frac{g}{a}U = 0$$

$$\Rightarrow \quad \boxed{\frac{d^2U}{dt^2} + \frac{g}{4a}U = 0}$$

$\Rightarrow$ équation differentielle de type $\ddot{x} + \omega^2 x = 0$

ou $\omega^2 = \frac{g}{4a} \Rightarrow \omega = \sqrt{\frac{g}{4a}}$, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4a}{g}}$.

Ex2 :

$$E_c = \sum_{i=1}^{N}\frac{1}{2}m_i v_i^2,$$

$$E_C = \frac{1}{2}\sum_{\alpha,\beta} m_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta$$ (en coordonnées généralisées)

$$P_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{q}_\alpha} = \frac{\partial(E_c - U)}{\partial\dot{q}_\alpha} = \frac{\partial E_c}{\partial\dot{q}_\alpha}$$ (on suppose U ne dépend pas de $\dot{q}_\alpha$)

$$P_\alpha = \sum_\beta m_{\alpha\beta}\dot{q}_\beta$$

$$H = \sum P_\alpha\dot{q}_\alpha - \mathcal{L} = \sum m_{\alpha\beta}\dot{q}_\beta\dot{q}_\alpha - \frac{1}{2}\sum m_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta + U$$

$$H = E_c + U = E.$$

Ex3 : $V = V(r)$, on peut montrer que la force est centrale : $\vec{F} = -\vec{\nabla}V(r) = -\frac{\partial V}{\partial r}\vec{e_r} - \underbrace{\frac{1}{r}\frac{\partial V}{\partial \theta}}_{0}\vec{e_\theta} - \underbrace{\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}}_{0}\vec{e_\varphi}$

$$\vec{F} = -\frac{\partial V}{\partial r}\vec{e_r}$$

$\vec{F}$ est portée sur $\vec{e_r}$ $\Rightarrow$ la force est centrale.

On a montré (chapitre 2) que lorsque la force est centrale le mouvement est plan. donc on va utiliser les coordonnées plaires (r, θ).

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \quad , q_\alpha = r, \theta$$

$$H = \sum p_\alpha \dot{q}_\alpha - \mathcal{L} \quad , p_\alpha = p_r , p_\theta$$

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}.$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$H = p_\theta \cdot \dot{\theta} + p_r \cdot \dot{r} - \mathcal{L} = p_\theta\left(\frac{p_\theta}{mr^2}\right) + p_r\left(\frac{p_r}{m}\right) - \frac{1}{2}m\left(\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2\right) + V(r)$$

$$\boxed{H(r, \theta, p_r, p_\theta) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)}$$

Equations de Hamilton :

$$\begin{cases} \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \\ \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \end{cases} \Rightarrow \begin{cases} \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \Rightarrow p_r = m\dot{r} \Rightarrow \dot{p}_r = m\ddot{r} \quad (1) \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \Rightarrow p_\theta = mr^2\dot{\theta} \quad (2) \\ \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \quad (3) \\ \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \Rightarrow p_\theta = K \quad (4) \end{cases}$$

$$(1) = (3) \Rightarrow \boxed{m\ddot{r} - \frac{K^2}{mr^3} + \frac{\partial V}{\partial r} = 0}$$

Ex4) : $\vec{\mathcal{L}} = \vec{r} \wedge \vec{P} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \wedge \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \begin{pmatrix} yP_z - zP_y \\ zP_x - xP_z \\ xP_y - yP_x \end{pmatrix}$

$$\begin{cases} \mathcal{L}_x = yP_z - zP_y \\ \mathcal{L}_y = zP_x - xP_z \\ \mathcal{L}_z = xP_y - yP_x \end{cases}$$

1) $[A, B] = \sum_{q_\alpha, P_\alpha} \left(\frac{\partial A}{\partial q_\alpha} \frac{\partial B}{\partial P_\alpha} - \frac{\partial A}{\partial P_\alpha} \frac{\partial B}{\partial q_\alpha} \right)$.

$$* \ [\mathcal{L}_x, \mathcal{L}_y] = \underbrace{\frac{\partial \mathcal{L}_x}{\partial x}}_{0} \frac{\partial \mathcal{L}_y}{\partial P_x} + \frac{\partial \mathcal{L}_x}{\partial y} \underbrace{\frac{\partial \mathcal{L}_y}{\partial P_y}}_{0} + \underbrace{\frac{\partial \mathcal{L}_x}{\partial z}}_{(-P_y)} \underbrace{\frac{\partial \mathcal{L}_y}{\partial P_z}}_{(-x)} - \underbrace{\frac{\partial \mathcal{L}_x}{\partial P_x}}_{0} \frac{\partial \mathcal{L}_y}{\partial x}$$

$$- \frac{\partial \mathcal{L}_x}{\partial P_y} \underbrace{\frac{\partial \mathcal{L}_y}{\partial y}}_{0} - \underbrace{\frac{\partial \mathcal{L}_x}{\partial P_z}}_{(y)} \cdot \underbrace{\frac{\partial \mathcal{L}_y}{\partial z}}_{(P_x)} = xP_y - yP_x = \mathcal{L}_z.$$

$$\Rightarrow [\mathcal{L}_x, \mathcal{L}_y] = \mathcal{L}_z.$$

et de la même façon $[\mathcal{L}_y, \mathcal{L}_z] = \mathcal{L}_x$ et $[\mathcal{L}_z, \mathcal{L}_x] = \mathcal{L}_y$.

$$* \ [P_x, \mathcal{L}_y] = \sum \left(\frac{\partial P_x}{\partial q_\alpha} \frac{\partial \mathcal{L}_y}{\partial P_\alpha} - \frac{\partial P_x}{\partial P_\alpha} \frac{\partial \mathcal{L}_y}{\partial q_\alpha} \right)$$

$$= \underbrace{\frac{\partial P_x}{\partial x}}_{0} \frac{\partial \mathcal{L}_y}{\partial P_x} + \underbrace{\frac{\partial P_x}{\partial y}}_{0} \cdot \frac{\partial \mathcal{L}_y}{\partial P_y} + \underbrace{\frac{\partial P_x}{\partial z}}_{0} \frac{\partial \mathcal{L}_y}{\partial P_z} - \overbrace{\frac{\partial P_x}{\partial P_x}}^{1} \overbrace{\frac{\partial \mathcal{L}_y}{\partial x}}^{-P_z} - \overbrace{\frac{\partial P_x}{\partial P_y}}^{0} \frac{\partial \mathcal{L}_y}{\partial y} - \overbrace{\frac{\partial P_x}{\partial P_z}}^{0} \frac{\partial \mathcal{L}_y}{\partial z}$$

$$\Rightarrow [P_x, \mathcal{L}_y] = P_z \quad \left(\frac{\partial P_x}{\partial x} = \frac{\partial P_x}{\partial y} = \frac{\partial P_x}{\partial z} = \frac{\partial P_x}{\partial P_y} = \frac{\partial P_x}{\partial P_z} = 0, \frac{\partial P_x}{\partial P_x} = 1 \right)$$

variables indépendants.

$$* \ [P_x, \mathcal{L}_x] = \overbrace{\frac{\partial P_x}{\partial P_x}}^{1} \overbrace{\frac{\partial \mathcal{L}_x}{\partial x}}^{0} = 0 \cdot 0 = 0.$$

Ex 5 : $H = \frac{p^2}{2m} + mgq$

$$H(q,p) = \frac{p^2}{2m} + mgq.$$

$$(q,p) \longmapsto (Q,P) : P = E = \frac{p^2}{2m} + mgq$$

$$Q = ?$$

pour que la transformation soit canonique

$$\Rightarrow [Q,P]_{p,q} = 1.$$

$$[Q,P] = \frac{\partial Q}{\partial q}\cdot\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\cdot\frac{\partial P}{\partial q} = \frac{\partial Q}{\partial q}\frac{p}{m} - \frac{\partial Q}{\partial p}\cdot mg = 1$$

$$\frac{\partial Q}{\partial q}\frac{p}{m} - \frac{\partial Q}{\partial p} mg = 1 \quad (1)$$

posant $Q(q,p,t) = f(q) + g(p) + h(t)$

$\frac{\partial Q}{\partial q} = \frac{\partial f}{\partial q}$, $\frac{\partial Q}{\partial p} = \frac{\partial g}{\partial p}$, on remplace dans (1)

$$\underbrace{\frac{\partial f(q)}{\partial q}}_{\text{dépend que de } q} = \underbrace{\frac{1}{p}\frac{\partial g(p)}{\partial p} m^2 g + \frac{m}{p}}_{\text{dépend que de } p} = K$$

$$\frac{\partial f}{\partial q} = K \Rightarrow f(q) = -\int K\,dq = Kq + C_1$$

$$\frac{\partial g}{\partial p} = \frac{p}{m^2 g}\left(K - \frac{m}{p}\right) \Rightarrow g(p) = \int\left(\frac{K}{m^2 g}p - \frac{1}{mg}\right)dp$$

$$= \frac{Kp^2}{2m^2 g} - \frac{p}{mg} + C_2$$

$$\boxed{Q(q,p,t) = Kq + \frac{Kp^2}{2m^2 g} - \frac{p}{mg} + C.}$$

4

Equation de Hamilton Jacobi. $H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + mgq + \frac{\partial S}{\partial t} = 0$$

on pose $S = S_1(q) + S_2(t)$

donc $\frac{\partial S}{\partial q} = \frac{\partial S_1}{\partial q}$ et $\frac{\partial S}{\partial t} = \frac{\partial S_2}{\partial t}$.

$$\Rightarrow \underbrace{\frac{1}{2m}\left(\frac{\partial S_1}{\partial q}\right)^2 + mgq}_{\text{dépend de } q} = \underbrace{-\frac{\partial S_2}{\partial t}}_{\text{dépend de } t} = \beta$$

$$\Rightarrow \frac{\partial S_2}{\partial t} = -\beta \Rightarrow S_2(t) = -\beta \int dt = -\beta t + c_1.$$

$$\frac{1}{2m}\left(\frac{\partial S_1}{\partial q}\right)^2 + mgq = \beta \Rightarrow S_1(q) = \pm \int \sqrt{2m(\beta - mgq)}\, dq.$$

$$S(q, \beta = P, t) = -\beta t \pm \frac{1}{3m^2 g}\left[2m(\beta - mgq)\right]^{\frac{3}{2}} + C.$$

les équations de transformations dérivent de deux équations

$$\begin{cases} p = \frac{\partial S}{\partial q} & \text{①} \\ Q = \frac{\partial S}{\partial P} \quad (\beta = P) & \text{②} \end{cases}$$

$$\text{①} \Rightarrow p = \pm \frac{1}{3m^2 g}(2m^2 g)\left[2m(P - mgq)\right]^{\frac{1}{2}}$$

$$\Rightarrow \boxed{P = \frac{p^2}{2m} + mgq}$$

$$\text{②} \Rightarrow \boxed{Q = \frac{\partial S}{\partial P} = -t - \frac{p}{mg}}$$

$$\Rightarrow \begin{cases} P = \frac{p^2}{2m} + mgq \\ Q = -t - \frac{p}{mg} \end{cases}$$

5