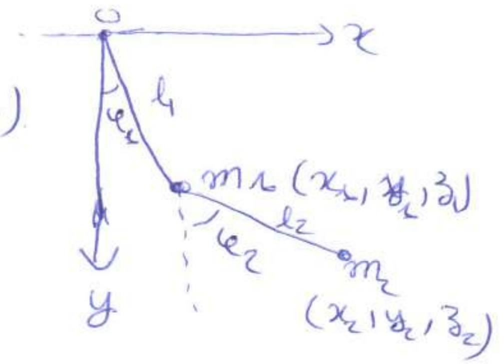
Série 2 :

Exercice 1 :



1) les équations des contraintes : (4 contraintes)

$z_1 = 0$

$z_2 = 0$

$x_1^2 + y_1^2 = l_1^2$

$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$.

le nombre de degrés de liberté $(N = 2)$, $(K = 4)$.

$n = 3N - K \Rightarrow n = 3 \times 2 - 4 = 2$.

$\Rightarrow$ on a 2 degrés de liberté.

$\Rightarrow$ les coordonnées généralisées $q_\alpha = \varphi_1, \varphi_2$.

2) $E_C = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2)$.

$$\begin{cases} x_1 = l_1 \sin\varphi_1 \\ y_1 = l_1 \cos\varphi_1 \\ x_2 = l_1 \sin\varphi_1 + l_2 \sin\varphi_2 \\ y_2 = l_1 \cos\varphi_1 + l_2 \cos\varphi_2 \\ z_1 = z_2 = 0 \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = \dot{\varphi}_1 l_1 \cos\varphi_1 \\ \dot{y}_1 = - \dot{\varphi}_1 l_1 \sin\varphi_1 \\ \dot{x}_2 = \dot{\varphi}_1 l_1 \cos\varphi_1 + \dot{\varphi}_2 l_2 \cos\varphi_2 \\ \dot{y}_2 = - l_1 \dot{\varphi}_1 \sin\varphi_1 - l_2 \dot{\varphi}_2 \sin\varphi_2 \end{cases}$$

Energie cinétique :

$$E_C = \frac{l_1^2}{2} m_1 \dot{\varphi}_1^2 + \frac{1}{2} m_2 (\dot{\varphi}_1^2 l_1^2 + \dot{\varphi}_2^2 l_2^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2))$$

Energie potentielle gravitationnelle :

$$U = - m_1 g y_1 - m_2 g y_2 = - m_1 g l_1 \cos\varphi_1 - m_2 g (l_1 \cos\varphi_1 + l_2 \cos\varphi_2)$$

$$= -(m_1 l_1 + m_2 l_2) g \cos\varphi_1 - m_2 g l_2 \cos\varphi_2$$

($\mathcal{L}$: le lagrangien).

$$\mathcal{L} = E_C - U$$

$$= \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 (l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2))$$

$$+ (m_1 l_1 + m_2 l_2) g \cos\varphi_1 + m_2 g l_2 \cos\varphi_2$$

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Les équations de Lagrange : $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0$

$$q_\alpha = \varphi_1, \varphi_2 \Rightarrow \begin{cases} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\varphi_1}\right) - \frac{\partial \mathcal{L}}{\partial \varphi_1} = 0 & (1) \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\varphi_2}\right) - \frac{\partial \mathcal{L}}{\partial \varphi_2} = 0 & (2) \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial \dot\varphi_1} = (m_1+m_2)\ell_1^2\dot\varphi_1 + m_2\ell_1\ell_2\dot\varphi_2\cos(\varphi_1-\varphi_2)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\varphi_1}\right) = (m_1+m_2)\ell_1^2\ddot\varphi_1 + m_2\ell_1\ell_2\ddot\varphi_2\cos(\varphi_1-\varphi_2) - m_2\ell_1\ell_2\dot\varphi_2(\dot\varphi_1-\dot\varphi_2)\sin(\varphi_1-\varphi_2)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_1} = -m_2\ell_1\ell_2\dot\varphi_1\dot\varphi_2\sin(\varphi_1-\varphi_2) - (m_1\ell_1+m_2\ell_2)g\sin\varphi_1$$

$$\Rightarrow (m_1+m_2)\ell_1^2\ddot\varphi_1 + m_2\ell_1\ell_2\ddot\varphi_2\cos(\varphi_1-\varphi_2) - m_2\ell_1\ell_2\dot\varphi_2(\dot\varphi_1-\dot\varphi_2)\sin(\varphi_1-\varphi_2) + m_2\ell_1\ell_2\dot\varphi_1\dot\varphi_2\sin(\varphi_1-\varphi_2) + (m_1\ell_1+m_2\ell_2)g\sin\varphi_1 = 0$$

$$\Rightarrow \boxed{(m_1+m_2)\ell_1^2\ddot\varphi_1 + m_2\ell_1\ell_2\ddot\varphi_2\cos(\varphi_1-\varphi_2) + m_2\ell_1\ell_2\dot\varphi_2^2\sin(\varphi_1-\varphi_2) + (m_1\ell_1+m_2\ell_2)g\sin\varphi_1 = 0} \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \dot\varphi_2} = m_2\ell_2^2\dot\varphi_2 + m_2\ell_1\ell_2\dot\varphi_1\cos(\varphi_1-\varphi_2)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\varphi_2}\right) = m_2\ell_2^2\ddot\varphi_2 + m_2\ell_1\ell_2\ddot\varphi_1\cos(\varphi_1-\varphi_2) - m_2\ell_1\ell_2\dot\varphi_1(\dot\varphi_1-\dot\varphi_2)\sin(\varphi_1-\varphi_2)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_2} = m_2\ell_1\ell_2\dot\varphi_1\dot\varphi_2\sin(\varphi_1-\varphi_2) - m_2 g\ell_2\sin\varphi_2$$

$$\Rightarrow \boxed{m_2\ell_2^2\ddot\varphi_2 + m_2\ell_1\ell_2\ddot\varphi_1\cos(\varphi_1-\varphi_2) - m_2\ell_1\ell_2\dot\varphi_1^2\sin(\varphi_1-\varphi_2) + m_2 g\ell_2\sin\varphi_2 = 0} \quad (2)$$

Approximation des petites angles : φ_1 et φ_2 sont petits $\Rightarrow$

$\cos(\varphi_1-\varphi_2) \simeq 1$, $\sin\varphi_1 = \varphi_1$, $\sin\varphi_2 = \varphi_2$, $\sin(\varphi_1-\varphi_2) = \varphi_1-\varphi_2 \simeq 0$

$$\Rightarrow \begin{cases} (m_1+m_2)\ell_1^2\ddot\varphi_1 + m_2\ell_1\ell_2\ddot\varphi_2 + (m_1\ell_1+m_2\ell_2)g\varphi_1 = 0 & (1) \\ m_2\ell_1\ell_2\ddot\varphi_1 + m_2\ell_2^2\ddot\varphi_2 + m_2 g\ell_2\varphi_2 = 0 & (2) \end{cases}$$

Ex2 : $\mathcal{L} = E_c - U$

en coordonnées sphériques $\vec{v} = \dot r\,\vec{e_r} + r\dot\theta\,\vec{e_\theta} + r\sin\theta\,\dot\varphi\,\vec{e_\varphi}$

$$E_c = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\varphi^2)$$

$$V = V(r,\theta).$$

$$\mathcal{L} = \frac{1}{2} m(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\varphi^2) - V(r,\theta).$$

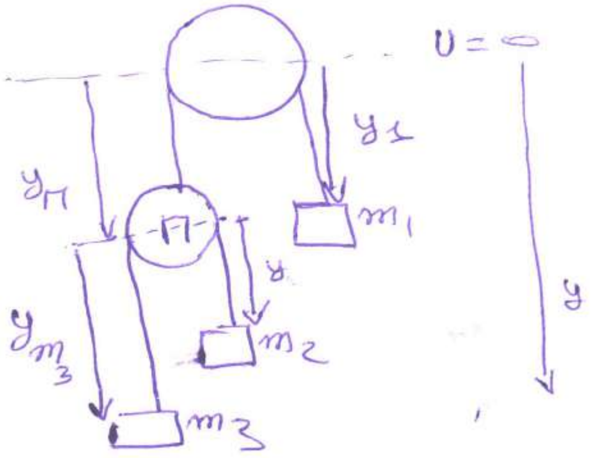
Ex 3 :

on a : $z_{m_1} = x_{m_1} = 0$

$z_{m_2} = x_{m_2} = 0$

$z_\Pi = x_\Pi = 0$

$z_{m_3} = x_{m_3} = 0$



$y_\Pi + y_1 = a \Rightarrow y_\Pi = a - y_1$

$y_2 + y_3 = b \Rightarrow y_3 = b - y_2$

on a 10 contraintes pour 4 masses : $(N = 4\ ;\ K = 10)$

$n = 3\times N - K = 3\times 4 - 10 = 2$.

on a 2 degrés de liberté : y_1 et y_2 coordonnées généralisées.

masse	y	vitesse	E_c	U
m_1	y_1	$\dot y_1$	$\frac{1}{2} m_1 \dot y_1^2$	$-m_1 g y_1$
Π	$a - y_1$	$-\dot y_1$	$\frac{1}{2}\Pi\,\dot y_1^2$	$-\Pi g(a - y_1)$
m_2	$y_2 + a - y_1$	$\dot y_2 - \dot y_1$	$\frac{1}{2} m_2 (\dot y_2 - \dot y_1)^2$	$-m_2 g(a + y_2 - y_1)$
m_3	$b - y_2 + a - y_1$	$-(\dot y_1 + \dot y_2)$	$\frac{1}{2} m_3 (\dot y_1 + \dot y_2)^2$	$-m_3 g(a + b - y_2 - y_1)$

Energie cinétique : $E_c = E_{c m_1} + E_{c\Pi} + E_{c m_2} + E_{c m_3}$

$$E_c = \frac{1}{2}(m_1 + \Pi)\dot y_1^2 + \frac{1}{2} m_2(\dot y_2 - \dot y_1)^2 + \frac{1}{2} m_3(\dot y_1 + \dot y_2)^2$$

Energie potentielle $U = U_{m_1} + U_\Pi + U_{m_2} + U_{m_3}$.

$$U = -g(m_1 - \Pi - m_2 - m_3)y_1 - (m_2 - m_3)g y_2 - (\Pi + m_2 + m_3)a - m_3 g b$$

$$\mathcal{L} = E_c - U = \frac{1}{2}(m_1+M)\dot{y}_1^2 + \frac{1}{2}m_2(\dot{y}_2-\dot{y}_1)^2 + \frac{1}{2}m_3(\dot{y}_1+\dot{y}_2)^2$$
$$+ g(m_1 - M - m_2 - m_3)y_1 + (m_2-m_3)g y_2 + (M+m_2+m_3)a + m_3 g b.$$

les équations de Lagrange.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0$$

ici $q_\alpha = y_1, y_2 \Rightarrow \begin{cases} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}_1}\right) - \frac{\partial \mathcal{L}}{\partial y_1} = 0 & (1) \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}_2}\right) - \frac{\partial \mathcal{L}}{\partial y_2} = 0 & (2) \end{cases}$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}_1} = (m_1+M)\dot{y}_1 - m_2(\dot{y}_2-\dot{y}_1) + m_3(\dot{y}_1+\dot{y}_2)$$
$$= (m_1+M+m_2+m_3)\dot{y}_1 + (m_3-m_2)\dot{y}_2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}_1}\right) = (m_1+M+m_2+m_3)\ddot{y}_1 + (m_3-m_2)\ddot{y}_2$$

$$\frac{\partial \mathcal{L}}{\partial y_1} = g(m_1-M-m_2-m_3)$$

$$\Rightarrow \boxed{(m_1+M+m_2+m_3)\ddot{y}_1 + (m_3-m_2)\ddot{y}_2 = g(m_1-M-m_2-m_3)} \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}_2} = m_2(\dot{y}_2-\dot{y}_1) + m_3(\dot{y}_1+\dot{y}_2)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}_2}\right) = (m_3-m_2)\ddot{y}_1 + (m_3+m_2)\ddot{y}_2$$

$$\frac{\partial \mathcal{L}}{\partial y_2} = g(m_2-m_3)$$

$$\boxed{(m_3-m_2)\ddot{y}_1 + (m_3+m_2)\ddot{y}_2 = g(m_2-m_3)} \quad (2)$$

(1) et (2) sont des équations du mouvement

$$\begin{cases} (m_1+M+m_2+m_3)\ddot{y}_1 + (m_3-m_2)\ddot{y}_2 = g(m_1-M-m_2-m_3) \\ (m_3-m_2)\ddot{y}_1 + (m_2+m_3)\ddot{y}_1 = g(m_2-m_3) \end{cases}$$

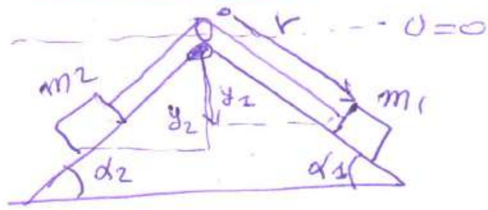
$$\ddot{y}_1 = \frac{\begin{vmatrix} (m_1 - \Pi - m_2 - m_3) g & (m_3 - m_2) \\ g(m_2 - m_3) & (m_2 + m_3) \end{vmatrix}}{\Delta}$$

$$\Delta = \begin{vmatrix} m_1 + \Pi + m_2 + m_3 & m_3 - m_2 \\ m_3 - m_2 & m_2 + m_3 \end{vmatrix}$$

$$\ddot{y}_2 = \frac{\begin{vmatrix} m_1 + \Pi + m_2 + m_3 & g(m_1 - \Pi - m_2 - m_3) \\ m_3 - m_2 & g(m_2 - m_3) \end{vmatrix}}{\Delta}$$

$$\ddot{y}_1 = \frac{(m_1 - \Pi)(m_2 + m_3) - 4 m_2 m_3}{(m_1 + \Pi)(m_2 + m_3) + 4 m_2 m_3}, \quad \ddot{y}_2 = \frac{2 g m_1 (m_2 - m_3)}{(m + \Pi)(m_2 + m_3) + 4 m_2 m_3}$$

pour que $\ddot{y}_1 = 0 \Longrightarrow \boxed{(m - \Pi)(m_2 + m_3) = 4 m_2 m_3}$



<u>EX4</u> : $om_1 \doteq r_1$, $om_2 = r_2$.

$om_1 + om_2 \doteq \ell$

$om_2 = \ell - om_1 = \ell - r \; (om_1 = r)$

donc on a 1 seul degré de liberté : coordonnée généralisée r.

$$E_c = \frac{1}{2} m_1 (\dot{r}^2) + \frac{1}{2} m_2 [(\ell - r)^\circ]^2 = \frac{1}{2} (m_1 + m_2) \dot{r}^2$$

$$U = - m_1 g y_1 - m_2 g y_2 = - m_1 g r \sin\alpha_1 - m_2 g (\ell - r) \sin\alpha_2$$

$$\mathcal{L} = E_c - U = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + g (m_1 \sin\alpha_1 - m_2 \sin\alpha_2) r + m_2 g \ell$$

Equation de Lagrange : $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0$

$q_\alpha = r$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{r}} = (m_1 + m_2) \dot{r} \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = (m_1 + m_2) \ddot{r}$$

$$\frac{\partial \mathcal{L}}{\partial r} = g (m_1 \sin\alpha_1 - m_2 \sin\alpha_2) \Longrightarrow (m_1 + m_2) \ddot{r} = g (m_1 \sin\alpha_1 - m_2 \sin\alpha_2)$$

$$\Rightarrow \ddot{r} = \frac{g (m_1 \sin\alpha_1 - m_2 \sin\alpha_2)}{m_1 + m_2}$$

equilibre $\Rightarrow \ddot{r} = 0 \Longrightarrow m_1 \sin\alpha_1 = m_2 \sin\alpha_2$

$$\boxed{\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{m_2}{m_1}}$$