Chapitre 4

PROGRAMMATION DYNAMIQUE

Introduction

La programmation dynamique est un paradigme de conception d'algorithmes permettant de résoudre les problèmes (d'optimisation en particulier) pouvant être décomposés en sous problèmes qui chevauchent et qui satisfont un critère dit principe d'optimalité de Bellman : si le problème peut être formulé comme un graphe multiétages (où la solution est une suite de décisions) alors une portion d'une solution optimale est elle aussi optimale. C'est une méthode exacte itérative considérée comme cas particulier du paradigme "diviser pour régner".

Principe :

On décompose le problème μa résoudre en sous problèmes plus simples (similaires au problem initial).

On trouve, récursivement, la solution des sous problèmes (sauf si le problème est trivial, auquel cas on calcule directement la solution).

On combine les solutions des sous problèmes pour obtenir la solution du problème initial.

Introduction

- inventée par le mathématicien américain Richard Bellman dans les années 50 pour la résolution des problèmes d'optimisation;

- programmation veut dire ici "planification"

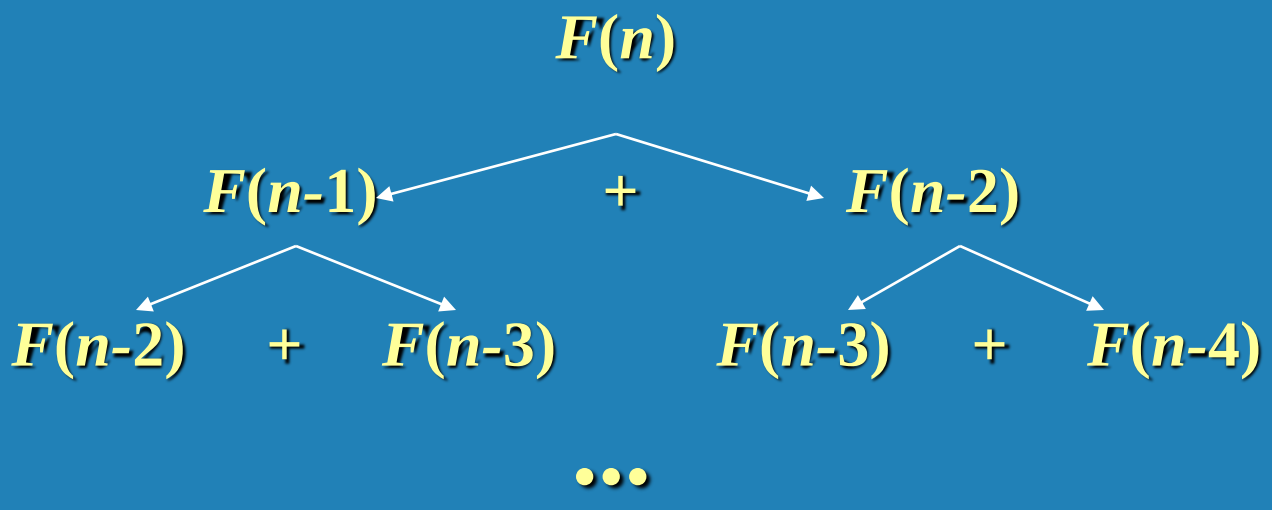
- démarche :
  - Etablir une équation de récurrence exprimant la solution d'un sous problème en fonction de celles de ses prédécesseurs (ou successeurs).
  - Résoudre les sous problèmes récursivement;
  - Stocker leurs solutions dans une table
  - Relever de cette table la solution du problème posé.

Exemple 1 : suite de Fibonacci

- **définition :**

$$F(n) = F(n\text{-}1) + F(n\text{-}2)$$
$$F(0) = 0$$
$$F(1) = 1$$

- **Computing the n^{th} Fibonacci number recursively (top-down):**



Exemple 1 : suite de Fibonacci

Computing the n^{th} Fibonacci number using bottom-up iteration and recording results:

$F(0) = 0$
$F(1) = 1$
$F(2) = 1+0 = 1$
…
$F(n\text{-}2) =$
$F(n\text{-}1) =$
$F(n) = F(n\text{-}1) + F(n\text{-}2)$

0	1	1	. . .	$F(n\text{-}2)$	$F(n\text{-}1)$	$F(n)$

Efficiency:
- **time** n
- **space** n

What if we solve it recursively?

Examples of DP algorithms

- Coefficient binomial
- Plus longue sous séquence commune
- Algorithme de warshall's pour la clôture transitive
- Algorithme de floyd's pour les plus courts chemins
- Construction d'un arbre binaire de recherche optimale
- Problème du sac à dos
- Problème du voyageur de commerce

Computing a binomial coefficient by DP

Binomial coefficients are coefficients of the binomial formula:

$$(a + b)^n = C(n,0)a^n b^0 + \ldots + C(n,k)a^{n-k}b^k + \ldots + C(n,n)a^0 b^n$$

Recurrence: $C(n,k) = C(n\text{-}1,k) + C(n\text{-}1,k\text{-}1)$ **for** $n > k > 0$

$$C(n,0) = 1, \quad C(n,n) = 1 \text{ for } n \geq 0$$

Value of $C(n,k)$ can be computed by filling a table:

	0	1	2	. . .	k-1	k
0	1					
1	1	1				
.						
.						
.						
n-1					$C(n\text{-}1,k\text{-}1)$	$C(n\text{-}1,k)$
n						$C(n,k)$

Computing $C(n,k)$: pseudocode and analysis

ALGORITHM *Binomial*(n, k)

```
//Computes C(n, k) by the dynamic programming algorithm
//Input: A pair of nonnegative integers n ≥ k ≥ 0
//Output: The value of C(n, k)
for i ← 0 to n do
    for j ← 0 to min(i, k) do
        if j = 0 or j = i
            C[i, j] ← 1
        else C[i, j] ← C[i − 1, j − 1] + C[i − 1, j]
return C[n, k]
```

Time efficiency: $\Theta(nk)$

Space efficiency: $\Theta(nk)$

Knapsack Problem by DP

Given n items of

integer weights: $w_1 \quad w_2 \quad \dots \quad w_n$

values: $v_1 \quad v_2 \quad \dots \quad v_n$

a knapsack of integer capacity W

find most valuable subset of the items that fit into the knapsack

Consider instance defined by first i items and capacity j ($j \le W$).

Let $V[i,j]$ be optimal value of such an instance. Then

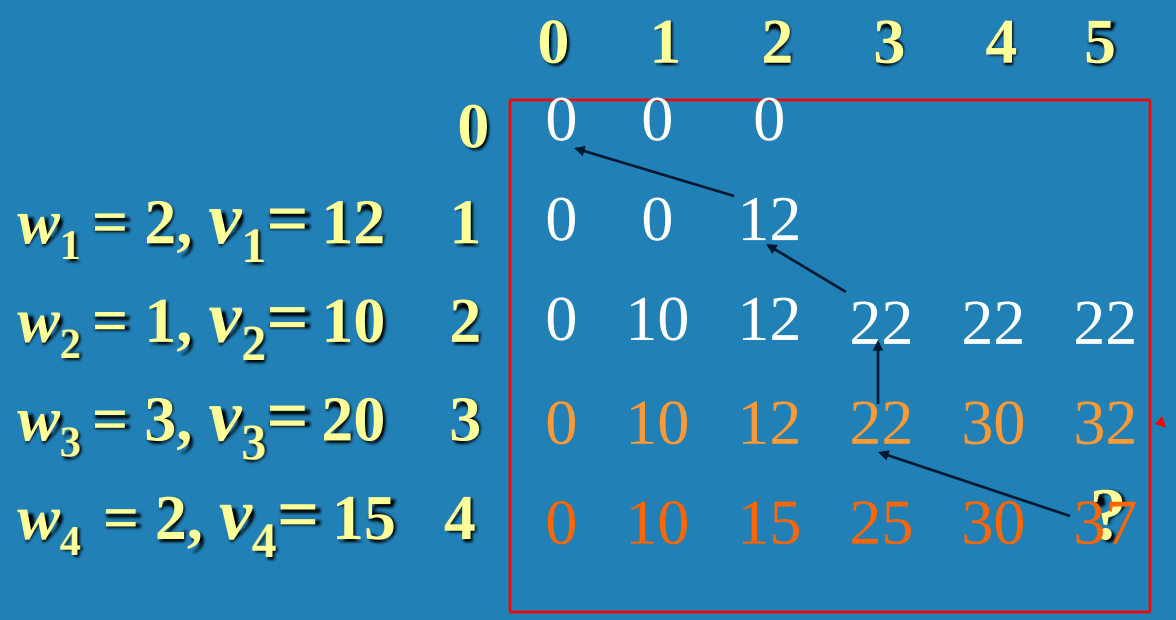
$$V[i,j] = \begin{cases} \max \{V[i-1,j],\ v_i + V[i-1,j-w_i]\} & \text{if } j-w_i \ge 0 \\ V[i-1,j] & \text{if } j-w_i < 0 \end{cases}$$

Initial conditions: $V[0,j] = 0$ and $V[i,0] = 0$

Knapsack Problem by DP (example)

Example: Knapsack of capacity $W = 5$

item	weight	value
1	2	$12
2	1	$10
3	3	$20
4	2	$15



Backtracing finds the actual optimal subset, i.e. solution.

Knapsack Problem by DP (pseudocode)

```
Algorithm DPKnapsack(w[1..n], v[1..n], W)
    var V[0..n,0..W],  P[1..n,1..W]: int
    for j := 0 to W do
        V[0,j] := 0
    for i := 0 to n do
        V[i,0] := 0
    for i := 1 to n do
        for j := 1 to W do
            if  w[i] ≤ j and v[i] + V[i-1,j-w[i]] > V[i-1,j] then
                V[i,j] := v[i] + V[i-1,j-w[i]]; P[i,j] := j-w[i]
            else
                V[i,j] := V[i-1,j]; P[i,j] := j
    return V[n,W] and the optimal subset by backtracing
```

Running time and space:
O(nW).

Longest Common Subsequence (LCS)

- A subsequence of a sequence/string *S* is obtained by deleting zero or more symbols from *S*. For example, the following are some subsequences of "president": pred, sdn, predent. In other words, the letters of a subsequence of S appear in order in *S*, but they are not required to be consecutive.

- The longest common subsequence problem is to find a maximum length common subsequence between two sequences.

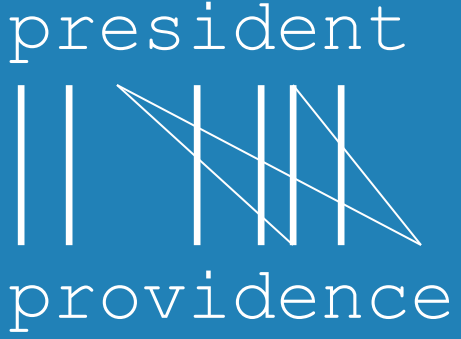
LCS

For instance,

Sequence 1: president

Sequence 2: providence

Its LCS is priden.



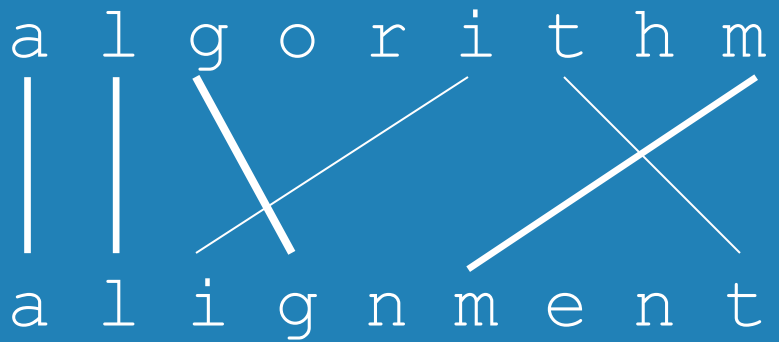
LCS

Another example:

Sequence 1: algorithm

Sequence 2: alignment

One of its LCS is algm.



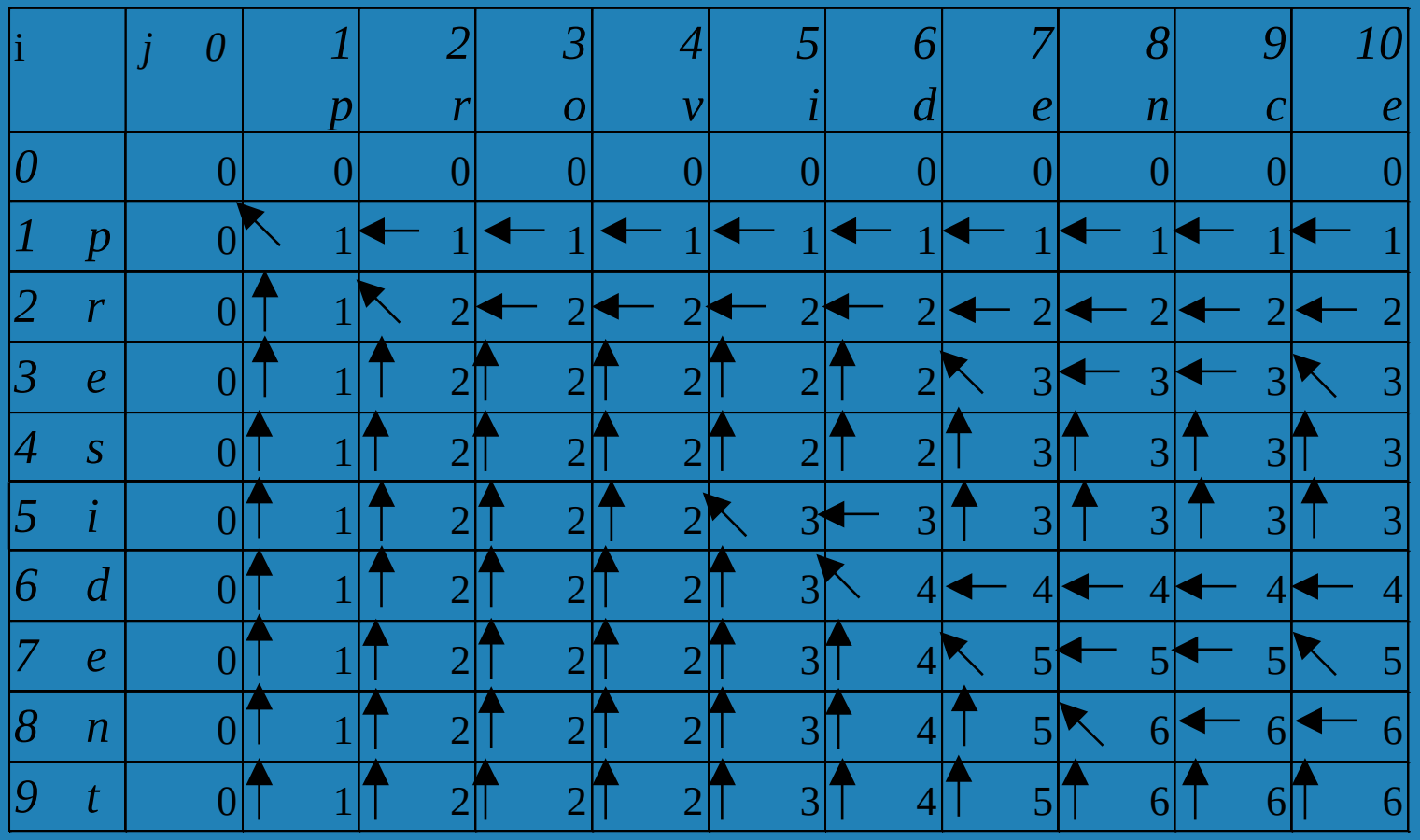
How to compute LCS?

- Let A=$a_1a_2\ldots a_m$ and $B=b_1b_2\ldots b_n$.
- *len*(*i*, *j*): the length of an LCS between
  $a_1a_2\ldots a_i$ and $b_1b_2\ldots b_j$
- With proper initializations, *len*(*i*, *j*) can be computed as follows.

$$len(i,j)=\begin{cases}0 & \text{if } i=0 \text{ or } j=0,\\ len(i-1,j-1)+1 & \text{if } i,j>0 \text{ and } a_i=b_j,\\ \max(len(i,j-1),len(i-1,j)) & \text{if } i,j>0 \text{ and } a_i\neq b_j.\end{cases}$$

procedure *LCS-Length(A, B)*

1. **for** $i \leftarrow 0$ **to** m **do** $len(i,0) = 0$
2. **for** $j \leftarrow 1$ **to** n **do** $len(0,j) = 0$
3. **for** $i \leftarrow 1$ **to** m **do**
4. **for** $j \leftarrow 1$ **to** n **do**
5. **if** $a_i = b_j$ **then** $\begin{cases} len(i,j) = len(i-1, j-1) + 1 \\ prev(i,j) = \text{"} \nwarrow \text{"} \end{cases}$
6. **else if** $len(i-1, j) \geq len(i, j-1)$
7. **then** $\begin{cases} len(i,j) = len(i-1, j) \\ prev(i,j) = \text{"} \uparrow \text{"} \end{cases}$
8. **else** $\begin{cases} len(i,j) = len(i, j-1) \\ prev(i,j) = \text{"} \leftarrow \text{"} \end{cases}$
9. **return** *len* and *prev*



Running time and memory: O(*mn*) and O(*mn*).

The backtracing algorithm

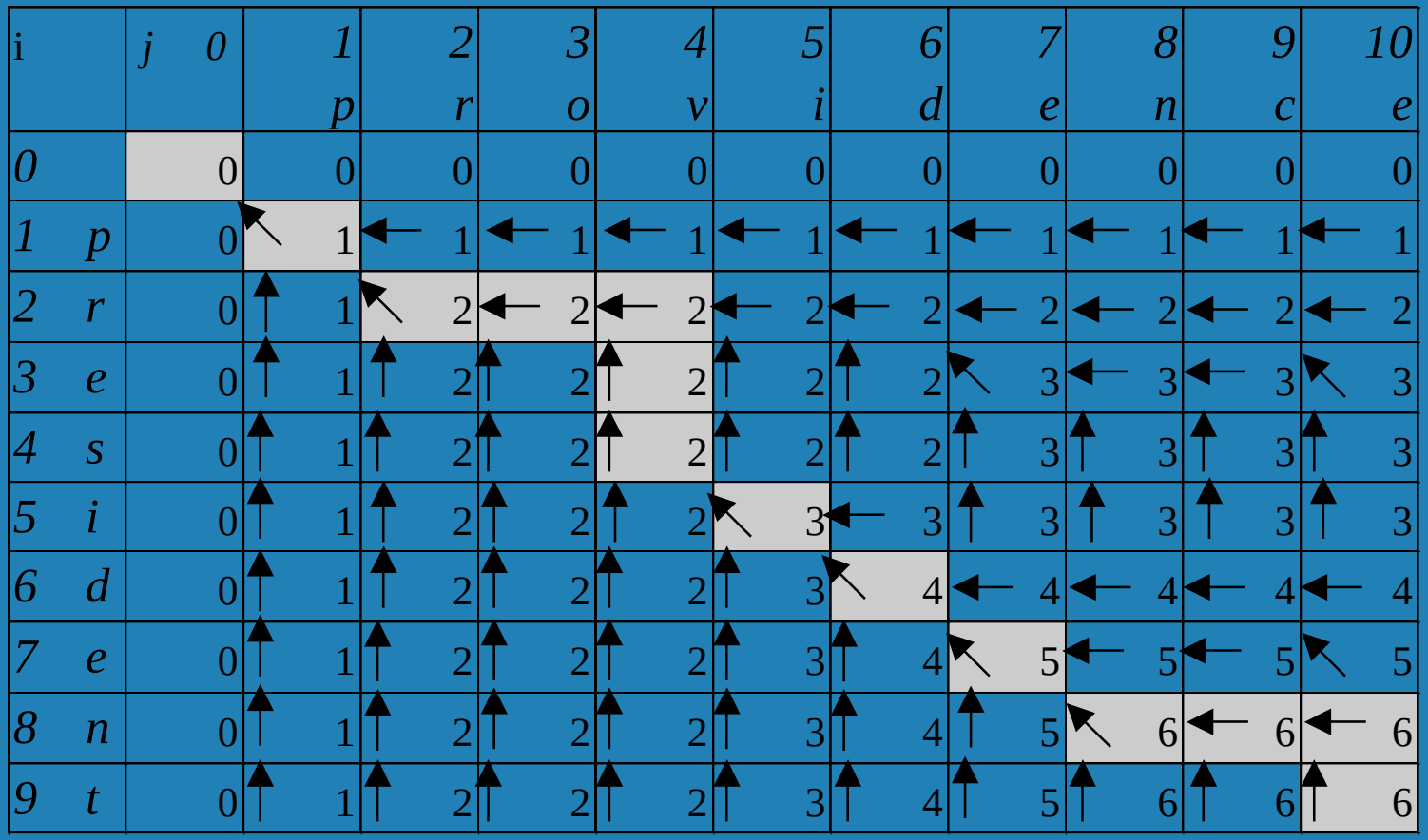
procedure *Output-LCS(A, prev, i, j)*

1 **if** $i = 0$ **or** $j = 0$ **then return**

2 **if** *prev(i, j)=”* ↖ *“* **then** $\begin{cases} Output - LCS(A, prev, i-1, j-1) \\ \text{print} \quad a_i \end{cases}$

3 **else if** *prev(i, j)=”* ↑ *“* **then** *Output-LCS(A, prev, i-1, j)*

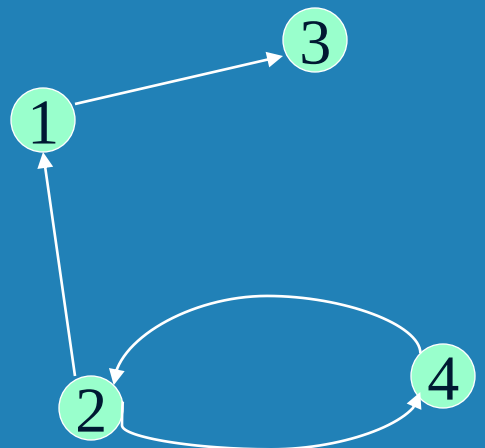
4 **else** *Output-LCS(A, prev, i, j-1)*



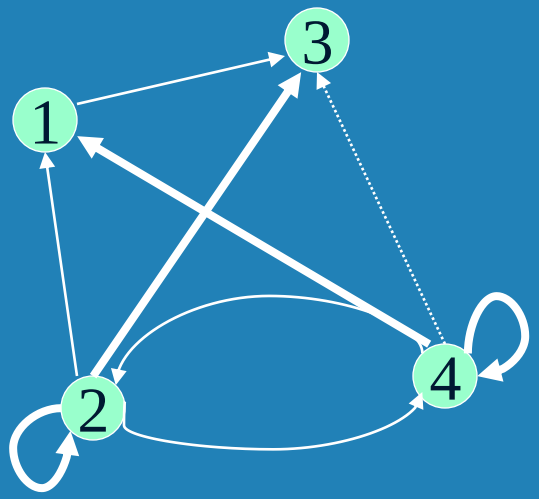
Output: *priden*

Warshall's Algorithm: Transitive Closure

- **Computes the transitive closure of a relation**
- **Alternatively: existence of all nontrivial paths in a digraph**
- **Example of transitive closure:**



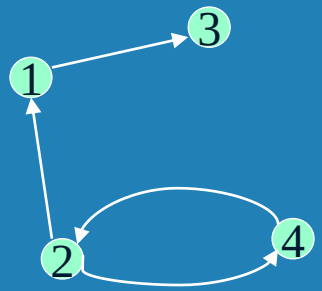
$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$



$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & \mathbf{1} & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} \end{matrix}$$

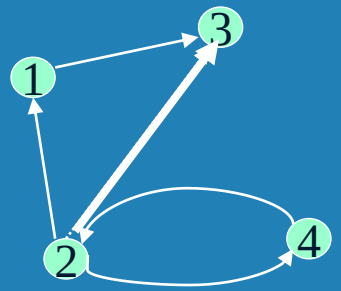
Warshall's Algorithm

Constructs transitive closure *T* as the last matrix in the sequence of *n*-by-*n* matrices $R^{(0)}, \ldots, R^{(k)}, \ldots, R^{(n)}$ where
$R^{(k)}[i,j] = 1$ iff there is nontrivial path from *i* to *j* with only the first *k* vertices allowed as intermediate
Note that $R^{(0)} = A$ (adjacency matrix), $R^{(n)} = T$ (transitive closure)



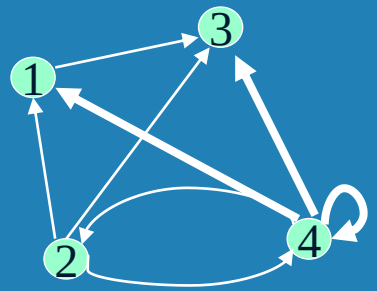
$R^{(0)}$

0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0



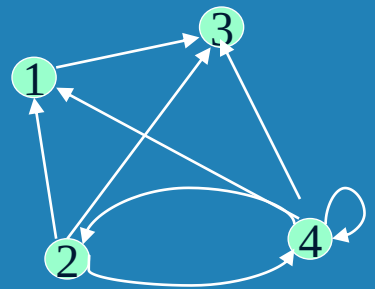
$R^{(1)}$

0	0	1	0
1	0	**1**	1
0	0	0	0
0	1	0	0



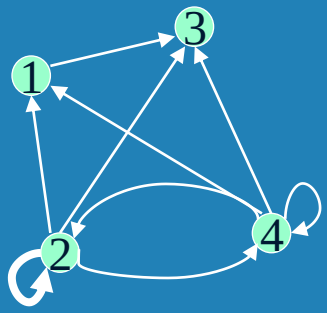
$R^{(2)}$

0	0	1	0
1	0	1	1
0	0	0	0
1	1	**1**	**1**



$R^{(3)}$

0	0	1	0
1	0	1	1
0	0	0	0
1	1	1	1



$R^{(4)}$

0	0	1	0
1	**1**	1	1
0	0	0	0
1	1	1	1

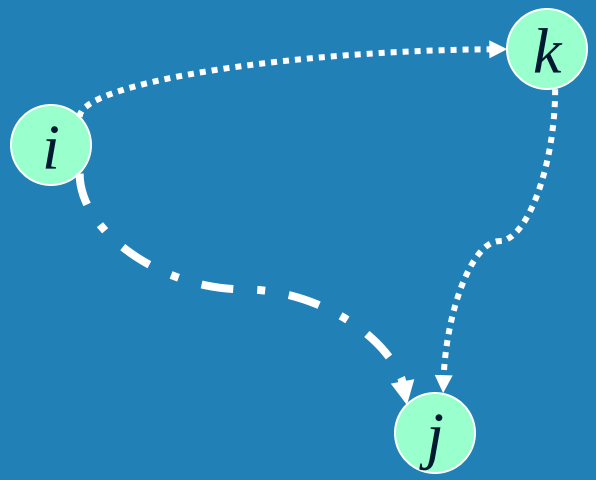
Warshall's Algorithm (recurrence)

On the *k*-th iteration, the algorithm determines for every pair of vertices *i, j* if a path exists from *i* and *j* with just vertices 1,…,*k* allowed as intermediate

$$R^{(k)}[i,j] = \begin{cases} R^{(k-1)}[i,j] \\ \text{or} \\ R^{(k-1)}[i,k] \text{ and } R^{(k-1)}[k,j] \end{cases}$$

(path using just 1 ,…,*k*-1)

(path from *i* to *k* and from *k* to *j* using just 1 ,…,*k*-1)



Initial condition?

Warshall's Algorithm (matrix generation)

Recurrence relating elements $R^{(k)}$ to elements of $R^{(k-1)}$ is:

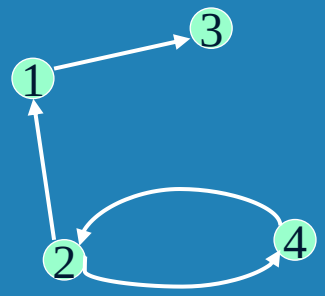
$$R^{(k)}[i,j] = R^{(k-1)}[i,j] \text{ or } (R^{(k-1)}[i,k] \text{ and } R^{(k-1)}[k,j])$$

It implies the following rules for generating $R^{(k)}$ from $R^{(k-1)}$:

Rule 1 If an element in row i and column j is 1 in $R^{(k-1)}$, it remains 1 in $R^{(k)}$

Rule 2 If an element in row i and column j is 0 in $R^{(k-1)}$, it has to be changed to 1 in $R^{(k)}$ if and only if the element in its row i and column k and the element in its column j and row k are both 1's in $R^{(k-1)}$

Warshall's Algorithm (example)



$R^{(0)} =$

0	0	1	0
1	0	0	1
0	0	0	0
0	1	0	0

$R^{(1)} =$

0	0	1	0
1	0	**1**	1
0	0	0	0
0	1	0	0

$R^{(2)} =$

0	0	1	0
1	0	1	1
0	0	0	0
1	1	**1**	**1**

$R^{(3)} =$

0	0	1	0
1	0	1	1
0	0	0	0
1	1	1	1

$R^{(4)} =$

0	0	1	0
1	**1**	1	1
0	0	0	0
1	1	1	1

Warshall's Algorithm (pseudocode and analysis)

ALGORITHM *Warshall*$(A[1..n, 1..n])$

```
//Implements Warshall's algorithm for computing the transitive closure
//Input: The adjacency matrix A of a digraph with n vertices
//Output: The transitive closure of the digraph
R^(0) ← A
for k ← 1 to n do
    for i ← 1 to n do
        for j ← 1 to n do
            R^(k)[i, j] ← R^(k−1)[i, j] or (R^(k−1)[i, k] and R^(k−1)[k, j])
return R^(n)
```

Time efficiency: $\Theta(n^3)$

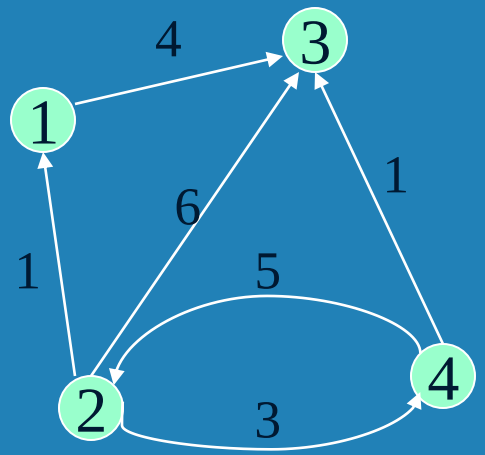
Space efficiency: Matrices can be written over their predecessors (with some care), so it's $\Theta(n^2)$.

Floyd's Algorithm: All pairs shortest paths

Problem: In a weighted (di)graph, find shortest paths between every pair of vertices

Same idea: construct solution through series of matrices $D^{(0)}$, …, $D^{(n)}$ using increasing subsets of the vertices allowed as intermediate

Example:

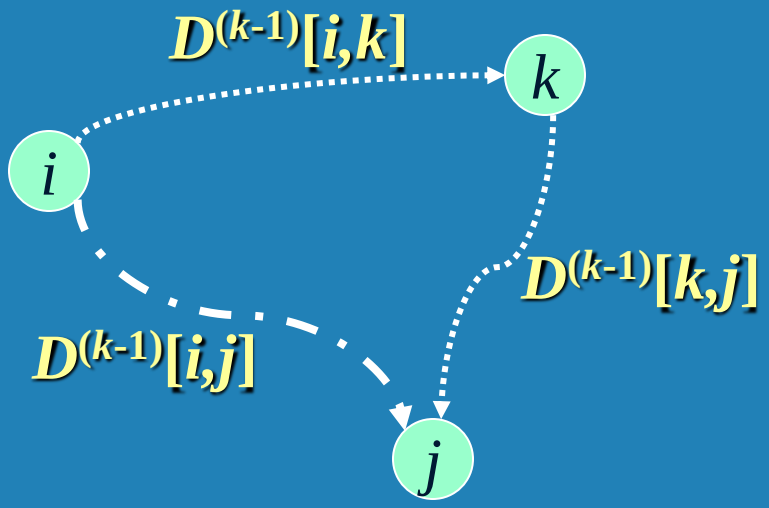


$$\begin{matrix} 0 & \infty & 4 & \infty \\ 1 & 0 & 4 & 3 \\ \infty & \infty & 0 & \infty \\ 6 & 5 & 1 & 0 \end{matrix}$$

Floyd's Algorithm (matrix generation)

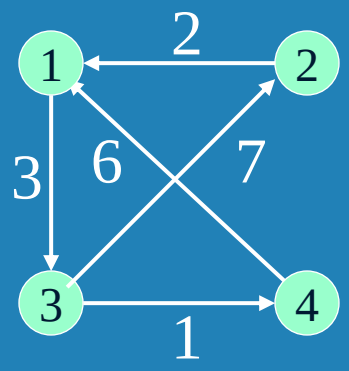
On the *k*-th iteration, the algorithm determines shortest paths between every pair of vertices *i, j* that use only vertices among 1,…,*k* as intermediate

$$D^{(k)}[i,j] = \min \{D^{(k-1)}[i,j],\ D^{(k-1)}[i,k] + D^{(k-1)}[k,j]\}$$



Initial condition?

Floyd's Algorithm (example)



$D^{(0)} =$

0	∞	3	∞
2	0	∞	∞
∞	7	0	1
6	∞	∞	0

$D^{(1)} =$

0	∞	3	∞
2	0	**5**	∞
∞	7	0	1
6	∞	**9**	0

$D^{(2)} =$

0	∞	3	∞
2	0	5	∞
9	7	0	1
6	∞	9	0

$D^{(3)} =$

0	**10**	3	**4**
2	0	5	**6**
9	7	0	1
6	**16**	9	0

$D^{(4)} =$

0	10	3	4
2	0	5	6
7	7	0	1
6	16	9	0

Floyd's Algorithm (pseudocode and analysis)

ALGORITHM *Floyd*$(W[1..n, 1..n])$

```
//Implements Floyd's algorithm for the all-pairs shortest-paths problem
//Input: The weight matrix W of a graph with no negative-length cycle
//Output: The distance matrix of the shortest paths' lengths
D ← W //is not necessary if W can be overwritten
for k ← 1 to n do
    for i ← 1 to n do
        for j ← 1 to n do
            D[i, j] ← min{D[i, j], D[i, k] + D[k, j]}
return D
```

If $D[i,k] + D[k,j] < D[i,j]$ then $P[i,j] \leftarrow k$

Time efficiency: $\Theta(n^3)$

Since the superscripts k or $k-1$ make no difference to $D[i,k]$ and $D[k,j]$.

Space efficiency: Matrices can be written over their predecessors

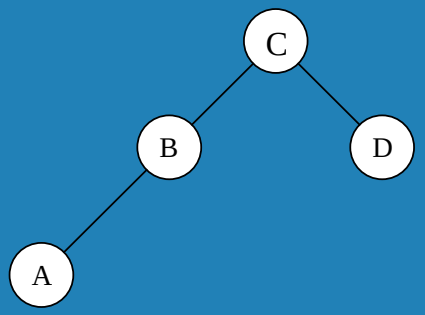
Note: Works on graphs with negative edges but without negative cycles. Shortest paths themselves can be found, too. How?

Optimal Binary Search Trees

Problem: Given n keys $a_1 < \ldots < a_n$ and probabilities $p_1, \ldots, p_n$ searching for them, find a BST with a minimum average number of comparisons in successful search.

Since total number of BSTs with n nodes is given by $C(2n,n)/(n+1)$, which grows exponentially, brute force is hopeless.

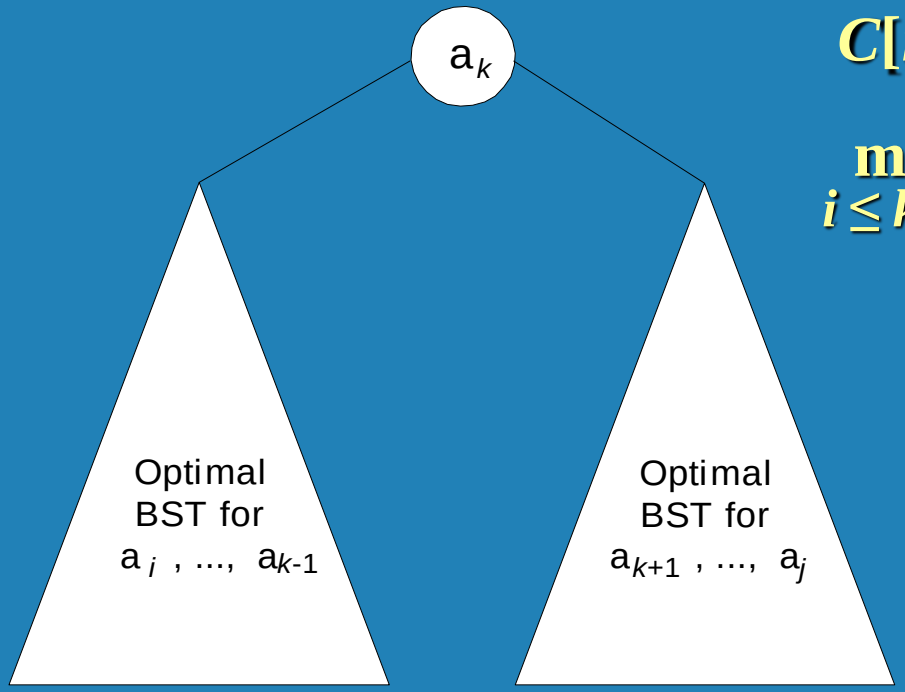
Example: What is an optimal BST for keys A, B, C, and D with search probabilities 0.1, 0.2, 0.4, and 0.3, respectively?



Average # of comparisons
= 1*0.4 + 2*(0.2+0.3) + 3*0.1
= 1.7

DP for Optimal BST Problem

Let $C[i,j]$ be minimum average number of comparisons made in $T[i,j]$, optimal BST for keys $a_i < \ldots < a_j$, where $1 \le i \le j \le n$. Consider optimal BST among all BSTs with some a_k $(i \le k \le j)$ as their root; $T[i,j]$ is the best among them.



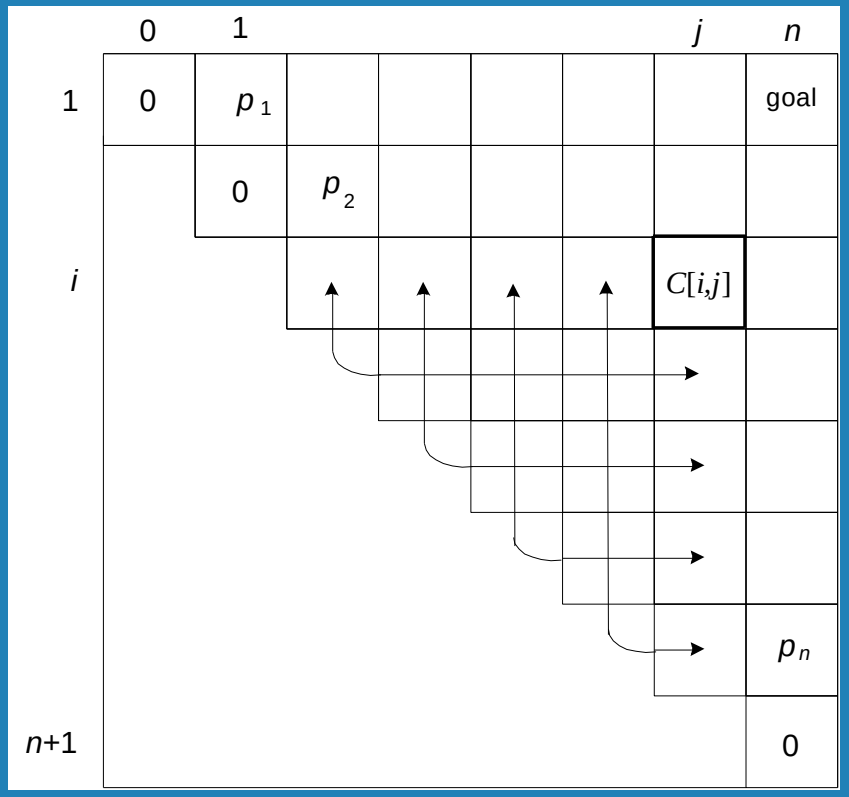
$$C[i,j] = \min_{i \le k \le j} \{p_k \cdot 1 + \sum_{s=i}^{k-1} p_s \text{ (level } a_s \text{ in } T[i,k-1] +1) + \sum_{s=k+1}^{j} p_s \text{ (level } a_s \text{ in } T[k+1,j] +1)\}$$

DP for Optimal BST Problem (cont.)

After simplifications, we obtain the recurrence for $C[i,j]$:

$$C[i,j] = \min_{i \le k \le j} \{C[i,k-1] + C[k+1,j]\} + \sum_{s=i}^{j} p_s \quad \text{for } 1 \le i \le j \le n$$

$$C[i,i] = p_i \quad \text{for } 1 \le i \le j \le n$$



Example: key A B C D

probability 0.1 0.2 0.4 0.3

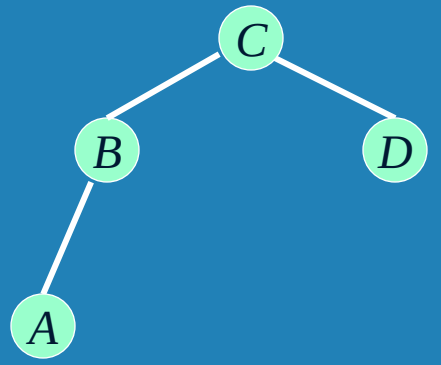
The tables below are filled diagonal by diagonal: the left one is filled using the recurrence

$$C[i,j] = \min_{i \le k \le j} \{C[i,k-1] + C[k+1,j]\} + \sum_{s=i}^{j} p_s, \quad C[i,i] = p_i;$$

the right one, for trees' roots, records k's values giving the minima

i \ j	0	1	2	3	4
1	0	.1	.4	1.1	1.7
2		0	.2	.8	1.4
3			0	.4	1.0
4				0	.3
5					0

i \ j	0	1	2	3	4
1		1	2	3	3
2			2	3	3
3				3	3
4					4
5					



optimal BST

Optimal Binary Search Trees

ALGORITHM *OptimalBST*$(P[1..n])$

```
//Finds an optimal binary search tree by dynamic programming
//Input: An array P[1..n] of search probabilities for a sorted list of n keys
//Output: Average number of comparisons in successful searches in the
//        optimal BST and table R of subtrees' roots in the optimal BST
for i ← 1 to n do
    C[i, i − 1] ← 0
    C[i, i] ← P[i]
    R[i, i] ← i
C[n + 1, n] ← 0
for d ← 1 to n − 1 do //diagonal count
    for i ← 1 to n − d do
        j ← i + d
        minval ← ∞
        for k ← i to j do
            if C[i, k − 1] + C[k + 1, j] < minval
                minval ← C[i, k − 1] + C[k + 1, j]; kmin ← k
        R[i, j] ← kmin
        sum ← P[i]; for s ← i + 1 to j do sum ← sum + P[s]
        C[i, j] ← minval + sum
return C[1, n], R
```

Analysis DP for Optimal BST Problem

Time efficiency: $\Theta(n^3)$ but can be reduced to $\Theta(n^2)$ by taking advantage of monotonicity of entries in the root table, i.e., $R[i,j]$ is always in the range between $R[i,j-1]$ and $R[i+1,j]$

Space efficiency: $\Theta(n^2)$

Method can be expanded to include unsuccessful searches