Exercise 1

Graph Coloring: Given a graph, color the vertices such that no adjacent vertices have the same color. Use the smallest possible number of colors.

Exercise 2

Maximal Satisfiability: Given a Boolean expression in CNF, find a truth assignment to the literals that satisfies as many of the clauses as possible.

Exercise 3

Integer Linear Programming: Given an objective function f(x0, x1, ..., xn) and a set of linear constraints of the form sum(ci * xi) <= k, search the set of integer values for x0, x1, ..., xn that satisfy the constraints to find values that maximize f.

Exercise 4

Longest Path: Given a graph, find the longest path (sequence of edges that touches each vertex at most once)

Exercise 5

Maximum Independent Set: Given a graph, find the largest possible set S of vertices such that none of the vertices in S are adjacent to each other.

Exercise 6

Maximum Clique: Given a graph, find the largest possible set S of vertices such that all of the vertices in S are adjacent to each other (ok, if you've done the previous one this is pretty trivial!). You may have noticed that a lot of these are optimization versions of classic NP-Complete problems. Let's continue that theme.

Exercise 7

Triangle Matching: Given a graph, find the maximum number of DISJOINT vertex sets {a, b, c} such that within each set, all the vertices are adjacent. By DISJOINT (which is naturally easier to understand because I am shouting), we mean that no vertex can belong to more than one of the sets.

Exercise 8

Minimum Vertex Cover: Given a graph, find the smallest possible set S of vertices such that every edge in the graph has at least one end in S.

Exercise 9

Travelling Repairperson Problem: Given a graph G with weighted edges and a start vertex s, find a traversal (not necessarily a path) of the graph that starts at s, visits each vertex at least once, and minimize the sum of the weights of the edges traversed before each vertex is reached for the first time. (If this seems confusing, the story behind the problem title should help - imagine that a repairperson needs to fix a machine at each vertex, and the weights on the edges represent the time it takes to travel along the edges. The waiting time for each vertex is the sum of the weights of all edges travelled before the vertex is reached for the first time. We want to minimize the total waiting time).

Exercise 10

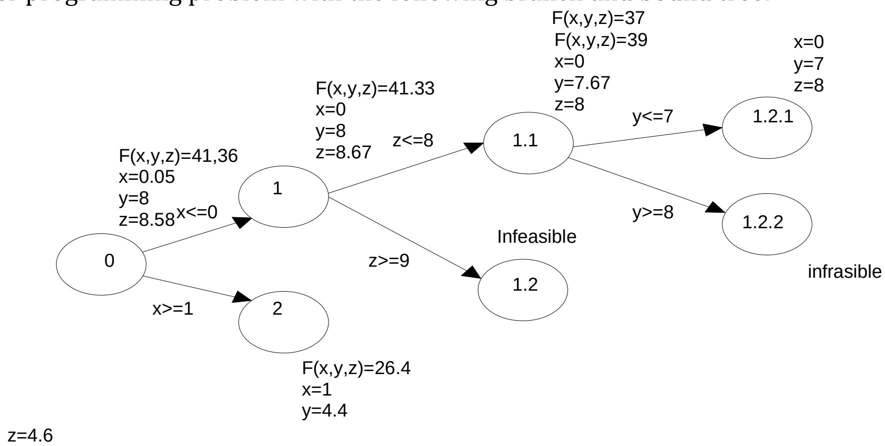
Wandering Salesperson Problem: Given a complete graph G with weighted edges with a start vertex s and a finishing vertex f, find a path that starts at s, finishes at f, visits every vertex exactly once, AND minimizes the largest edge weight used in the path. (Again, a story may help - if the weights represent travelling time, then we want to minimize the longest time that the salesperson spends on the road between stops.)

Exercise 11

Graph Partitioning: Given a graph G and an integer k, partition the vertices of G into k sets such that the number of edges that have their ends in different sets is minimized. This problem has applications to distributed computing. If the vertices represent processes and the edges represent shared data, and we have k processors in the network, then we may want to assign processes to processors in such a way that every processor is busy, and the sharing of data between processors is minimized.

Exercise 12

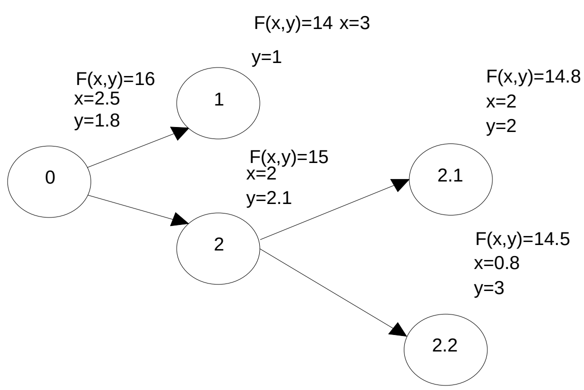
Given a certain integer liner programming problem with the following branch and bound tree:



a) Reason out if the objective of the problem is to maximize or minimize.
b) Have we reached the optimal solution? If we have, explain why, else say from which node we should branch and with which constraints.
c) Write the problem solved in node 1.2.1.

Exercise 13

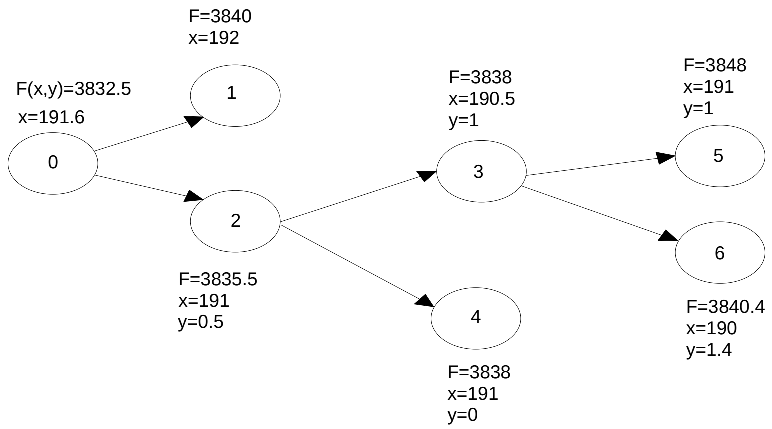
Given a certain integer linear programming problem with the following branch-and-bound tree:



a) Write the constraint added at each branch.
b) Have we reached the optimal solution? If so, explain why, else say from which node we should branch and with which constraints.

Exercise 14

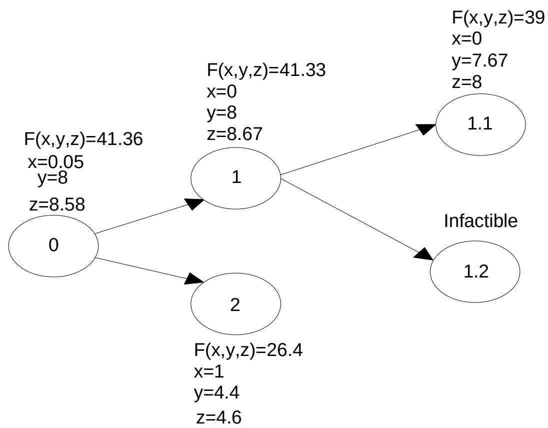
Let us consider an integer linear programming problem for which the following data of its branch-and-bound tree are known:



a) Reason out if it is a maximization or minimization problem.
b) Have we reached the optimal solution? If we have, explain why, else say from which node we should branch and with which constraints.
c) Write the problem solved in node 6.

Exercise 15

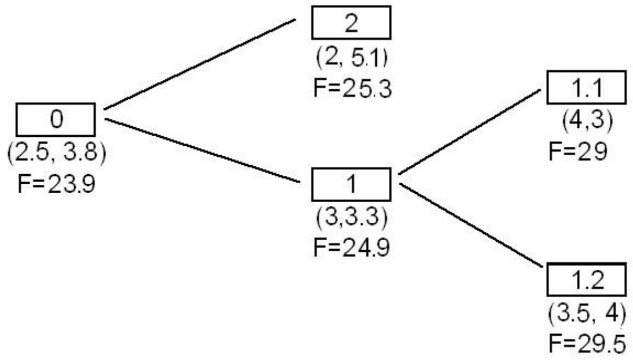
Given a certain mixed integer linear programming problem, where variables x and z have integrality condition, with the following branch-and-bound tree:



a) Reason out if it is a maximization or minimization problem.
b) Have we reached the optimal solution? If we have, explain why, else say from which node we should branch and with which constraints.
c) Write the problem solved in node 1.2.

Exercise 16

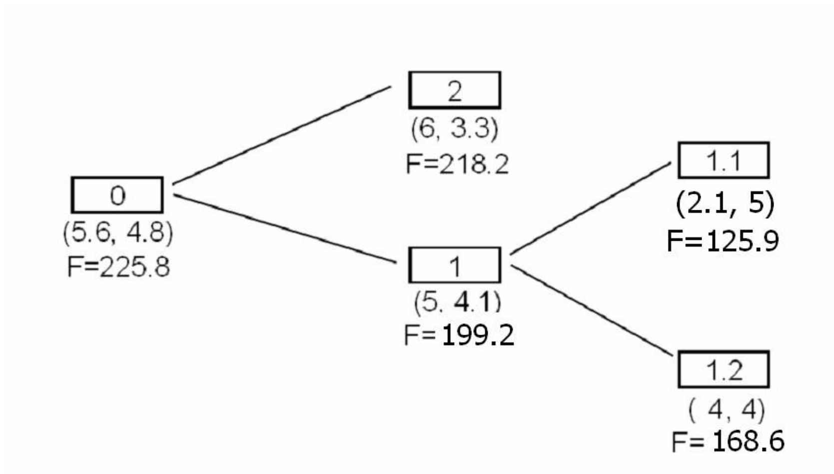
Let us consider an ILP problem with two variables x and y. Let us suppose we have the following associated branch-and-bound tree:



a) Reason out if it is a maximization or minimization problem.
b) Write the constraints that have been added at each branch.
c) Reason out if the optimal solution has already been found. If not, branch from the appropriate node.
d) If the original problem was a mixed ILP one where only variable x had to be integer, what would be the answer to question (c)?

Exercise 17

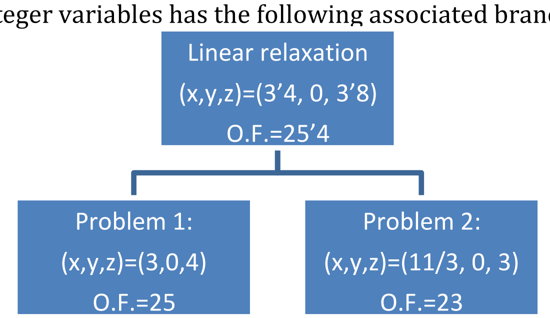
Let us consider an ILP problem with two variables x and y. Let us suppose we have the following associated branch-and-bound tree:



a) Reason out if it is a maximization or minimization problem.
b) Write the constraints that have been added at each branch.
c) Reason out if the optimal solution has already been found. If not, branch from the appropriate node.
d) If the original problem was a mixed ILP one where only variable x had to be integer, what would be the answer to question (c)?

Exercise 18

A linear problem with three integer variables has the following associated branch-and-bound tree:



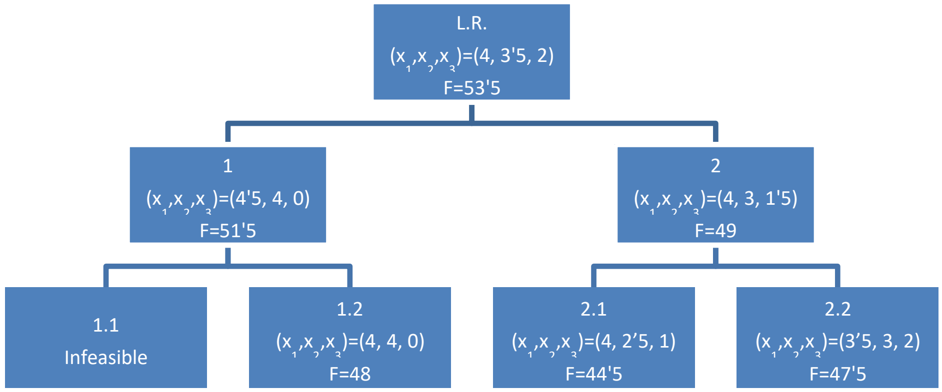
a) Write the constraint that has been added at each branch.

Check the right option :

- The optimal solution cannot be obtained yet: we have to branch on node 2.
- The optimal solution is (3, 0, 4) with O.F.=25
- The optimal solution is (11/3, 0, 3) with O.F.=23
- The optimal solution is (4, 0, 4) with O.F. =28

Exercise 19

The following solutions have been obtained in an integer linear programming problem:



a) Write the constraints that have been added at each branch.
b) Reason out if the optimal solution has been found. If it has been, say which one it is, else say form which problem we should branch and with which constraints.

Exercise 20

Let us consider the following mixed integer programming problem:

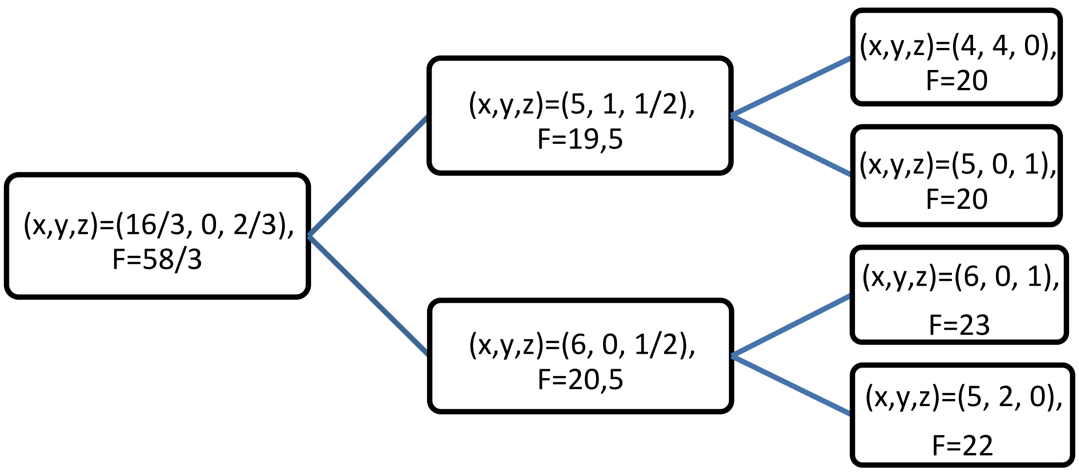
$$Min.\ F = 3x_1 + 2x_2 + 5x_3$$
$$s.t.\quad 2x1 + x2 + 2x3 \geq 12$$
$$x1 + x2 + 4x3 \geq 8$$
$$x1, x2, x3 \geq 0$$
$$x3 \text{ integer}$$

If the first problem solved when applying the branch-and-bound method provides the optimal solution $(x_1,x_2,x_3) = (16\ 3, 0, 2\ 3)$, with $F = 58/3$, check the box(es) that make sense for the solutions of the subproblems obtained after branching for the first time:

- $(x_1,x_2,x_3) = (6,0,12)$ with F=20'5 and $(x_1,x_2,x_3) = (5,1,12)$ with F=19'5
- $(x_1,x_2,x_3) = (4,4,0)$ with F=20 and $(x_1,x_2,x_3) = (5, 0,1)$ with F=20
- $(x_1,x_2,x_3) = (6, 0, 0)$ with F=18 and $(x_1,x_2,x_3) = (4,1,1)$ with F=19
- $(x_1,x_2,x_3) = (3,4,1)$ with F=22 and $(x_1,x_2,x_3) = (6,1, 0)$ with F=22

Exercise 21

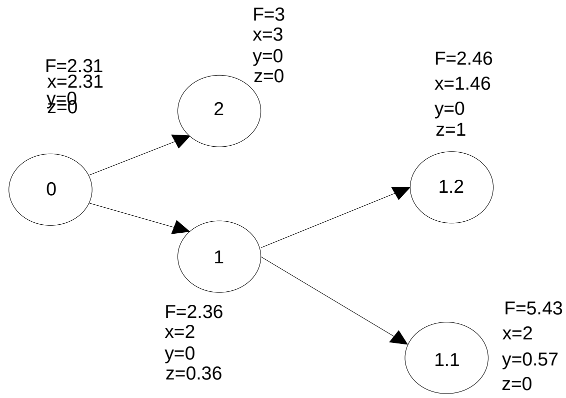
Let us consider the following branch-and-bound tree corresponding to an integer linear programming problem:



a) Add the constraint corresponding to each branch.
b) Reason out what the optimal solution is, and if it is a global maximum or minimum.
c) Determine which of the seven solutions seen in the tree is wrong and why.

Exercise 22

When solving the following integer linear programming problem, the following branch-and-bound tree is obtained:



$$? \quad x + 6y + z$$
$$s.t. \quad 13x + 7y + 11z \geq 30$$
$$x, y, z \in \mathbb{Z}^+$$

a) Reason out if it is a maximization or minimization problem and write the constraint associated with each branch.

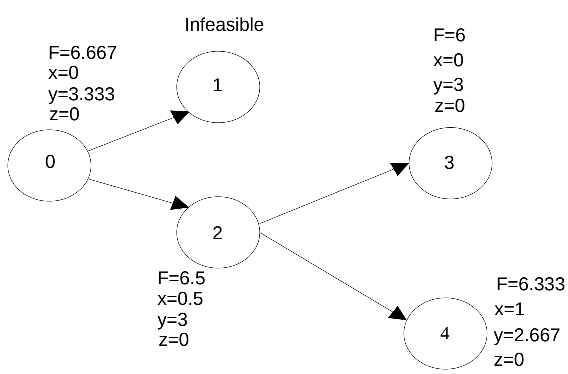
b) Have we reached the optimal solution? If so, explain why and point out which one it is. Else write the next branches and their associated constraints.

c) Consider the same branch-and-bound tree for the mixed integer problem with x, y $\in$Z+, z $\geq$0. Have we reached the optimal solution? If so, explain why and point out which one it is. Else write the next branches and their associated constraints.

Exercise 23

When solving the following integer linear programming problem, the following branch-and-bound tree is obtained:

$$? \quad x + 2y - 5z$$
$$s.t. \quad 2x + 3y + z \leq 10$$
$$x, y, z \in \mathbb{Z}^+$$



a) Reason out if it is a maximization or minimization problem and write the constraint associated with each branch.

b) Have we reached the optimal solution? If so, explain why and point out which one it is. Else write the next branches and their associated constraints, as well as the subproblems of the new nodes.

c) Consider the same branch-and-bound tree for the mixed integer problem with y, z $\in$Z+, x $\geq$0. Have we reached the optimal solution? If so, explain why and point out which one it is. Else write the next branches and their associated constraints.

Exercise 24

Given the following LP problem: Max. $x + 4y$

$$\text{s.t.} \quad -x + y \leq 2$$
$$2x + 3y \leq 12$$
$$2x + y \leq 8 \qquad x, y \geq 0$$

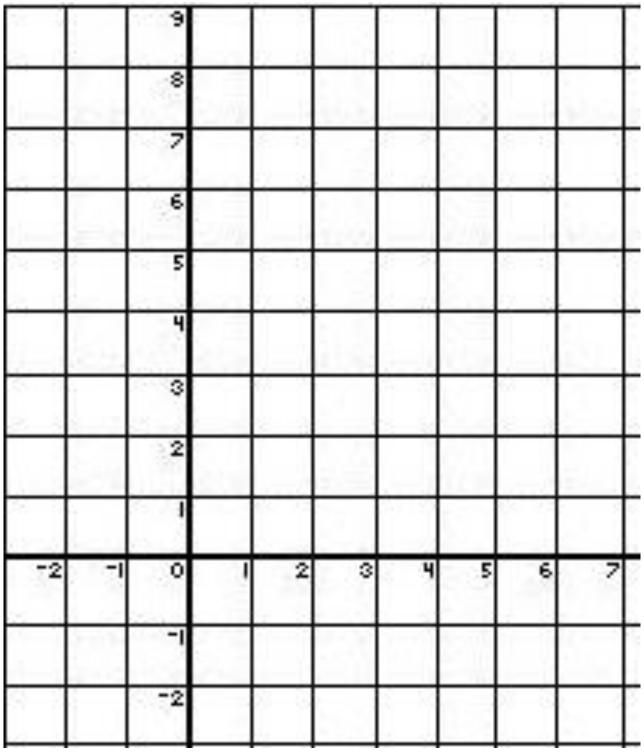
(a) Draw the solution set in the grid below.

(b) Obtain the optimal solution graphically.

(c) Consider the previous linear problem but with both variables integer.
Draw the solution set corresponding to this ILP.

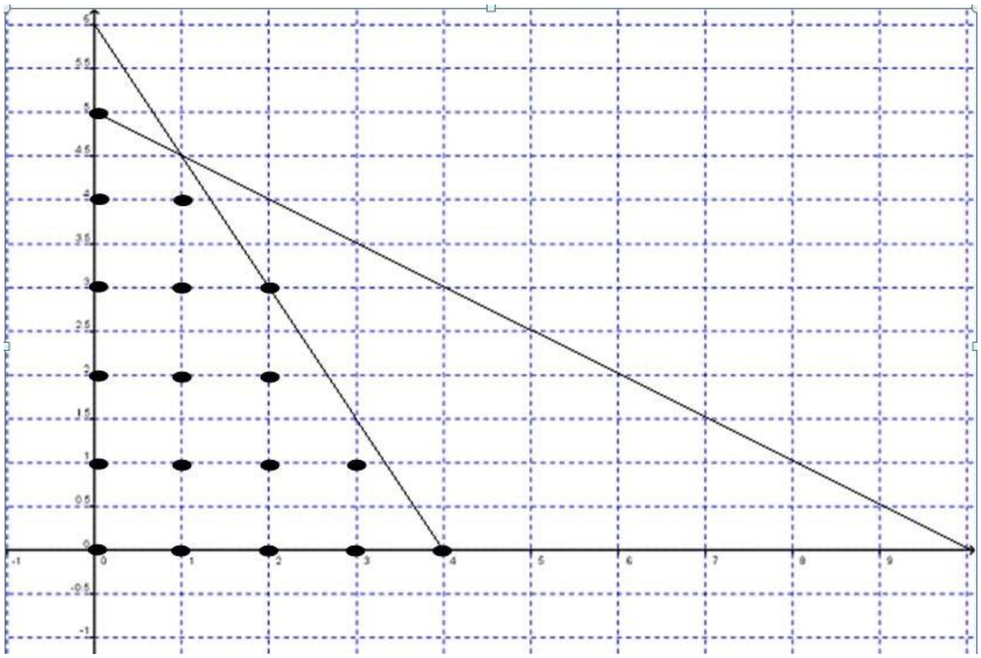
(d) Obtain the optimal solution of the ILP graphically.

(e) If we used the branch-and-bound method to solve the ILP, what would be the first branching that we would do from the root node?
Which problems would we solve in each of the new nodes?



Exercise 25

Let us suppose that we are solving a maximization ILP with the following solution set and the objective function 2x+2y.



Solve the problem applying the branch-and-bound method.

Exercise 26

We want to solve the following Integer Linear Programming problem

Min $-11x - y$

s.t. $10x+6y \leq 25$

$y \leq 2.5$ $x, y \in Z^+$

by means of its branch-and-bound tree. In order to do that, we have the optimal solutions of some LP problems obtained from the linear problem associated with our ILP and adding some constraints. The following tables show the optimal solutions of these LP's. Draw the tree using the information you need from the tables, adding the constraints on each branch, and explaining why you prune each node. When you do not have enough information to continue the tree, say if the optimal solution of the ILP has been found. If it has not been found, point out which node(s) you would have to branch.

Node 0 of the tree: The solution of the associated LP is (2.5,0), with f* = - 27.5.

Information regarding the nodes in the first level of the tree:

Added constraint	x ≥ 2	x ≥ 3	x ≤ 3	x ≤ 2
Optimal solution of the LP	(2.5,0), f* = - 27.5	Infeasible	(2.5,0), f* = - 27.5	(2,0.833), f* = -22.83

Information regarding the nodes in the second level of the tree (each cell contains the solution of the LP after adding the constraints in the corresponding row and column):

Added constraints	y ≤ 0	y ≤ 1	y ≥ 0	y ≥ 1
x ≥ 2	(2.5,0), f* = -27.5	(2.5,0), f* = -27.5	(2.5,0), f* = -27.5	(2.5,0), f* = -27.5
x ≥ 3	Infeasible	Infeasible	Infeasible	Infeasible
x ≤ 3	(2.5,0), f* = -27.5	(2.5,0), f* = -27.5	(2.5,0), f* = -27.5	(2.5,0), f* = -27.5
x ≤ 2	(2,0), f* = - 22	(2,0.833), f* = -22.83	(2,0.833), f* = -22.83	(1.9,1), f* = -21.9