Exercice 1

$$\forall\left((x,y,z)\,,\ (x_1,y_1,z_1)\right) \in \mathbb{R}^3\times\mathbb{R}^3,\quad (x,y,z)+(x_1,y_1,z_1)=(x+x_1,y+y_1,z+z_1)$$

$$\forall\alpha \in \mathbb{R},\ \forall\,(x,y,z)\in\mathbb{R}^3,\ \ \alpha\,(x,y,z)=(\alpha x,\alpha y,\alpha z)$$

a) ***Conditions sur la loi interne $+$ définie de $\mathbb{R}^3$ vers $\mathbb{R}^3$:*** $(\mathbb{R}^3,+)$ ***est un groupe commutatif***

a.1) ***la loi $+$ est commutative sur $\mathbb{R}^3$:***

Pour tout $u_1=(x_1,y_1,z_1)\in\mathbb{R}^3$, $u_2=(x_2,y_2,z_2)\in\mathbb{R}^3$, ***on a :***

$$u_1+u_2=(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1+z_2)=(x_2+x_1,y_2+y_1,z_2+z_1)$$
$$=(x_2,y_2,z_2)+(x_1,y_1,z_1)=u_2+u_1$$

a.2) ***la loi $+$ est associative sur $\mathbb{R}^3$:***

Pour tout $u_1=(x_1,y_1,z_1)\in\mathbb{R}^3$, $u_2=(x_2,y_2,z_2)\in\mathbb{R}^3$, $u_3=(x_3,y_3,z_3)\in\mathbb{R}^3$, ***on a :***

$$(u_1+u_2)+u_3=[(x_1,y_1,z_1)+(x_2,y_2,z_2)]+(x_3,y_3,z_3)=(x_1+x_2,y_1+y_2,z_1+z_2)+(x_3,y_3,z_3)$$
$$=((x_1+x_2)+x_3,(y_1+y_2)+y_3,(z_1+z_2)+z_3)$$
$$=(x_1+(x_2+x_3)\,,y_1+(y_2+y_3)\,,z_1+(z_2+z_3))$$
$$=(x_1,y_1,z_1)+(x_2+x_3,y_2+y_3,z_2+z_3)$$
$$=(x_1,y_1,z_1)+[(x_2,y_2,z_2)+(x_3,y_3,z_3)]=u_1+(u_2+u_3)$$

a.3) ***$\mathbb{R}^3$ admet un element neutre $0_{\mathbb{R}^3}$ par rapport à $+$:***

Pour $e=(e_1,e_2,e_3)\in\mathbb{R}^3$ ***et*** $(x,y,z)\in\mathbb{R}^3\Rightarrow(e_1,e_2,e_3)+(x,y,z)=(x,y,z)\Rightarrow$

$$\Rightarrow(e_1+x,e_2+y,e_3+z)=(x,y,z)\Rightarrow\begin{cases}e_1+x=x\\ e_2+y=y\\ e_3+z=z\end{cases}\Rightarrow\begin{cases}e_1=0\\ e_2=0\\ e_3=0\end{cases}\Rightarrow 0_{\mathbb{R}^3}=(0,0,0)\in\mathbb{R}^3$$

a.4) ***Chaque element $u=(x,y,z)\in\mathbb{R}^3$ admet un element symetrique $u'\in\mathbb{R}^3$:***

Pour $u=(x,y,z)$ ***et*** $u'=(x',y',z')$ ***on a :***

$$u+u'=0_{\mathbb{R}^3}\Leftrightarrow(x,y,z)+(x',y',z')=(0,0,0)\Leftrightarrow(x+x',y+y',z+z')=(0,0,0)$$
$$\Leftrightarrow\begin{cases}x+x'=0\\ y+y'=0\\ z+z'=0\end{cases}\Leftrightarrow\begin{cases}x'=-x\\ y'=-y\\ z'=-z\end{cases}\Leftrightarrow-(x,y,z)=-u=u'=(x',y',z')=(-x,-y,-z)\in\mathbb{R}^3$$

b) ***Conditions sur la loi externe définie de $(\mathbb{R},\mathbb{R}^3)$ vers $\mathbb{R}^3$:***

Pour tout $\alpha\in\mathbb{R},\beta\in\mathbb{R}, u=(x,y,z)\in\mathbb{R}^3$ ***et*** $v=(x',y',z')\in\mathbb{R}^3$

b.1) $(\alpha\beta)u=\alpha(\beta u)$

$$(\alpha\beta)u=(\alpha\beta)\,(x,y,z)=((\alpha\beta)x,(\alpha\beta)y,(\alpha\beta)z)\ =(\alpha\,(\beta x)\,,\alpha\,(\beta y)\,,\alpha\,(\beta z))$$
$$=\alpha\,(\beta x,\beta y,\beta z)=\alpha\,(\beta\,(x,y,z))=\alpha(\beta u)$$

b.2) $\underline{(\alpha+\beta)u=\alpha u+\beta u}$

$$(\alpha+\beta)u=(\alpha+\beta)(x,y,z)=((\alpha+\beta)x,(\alpha+\beta)y,(\alpha+\beta)z)$$

$$=(\alpha x+\beta x,\alpha y+\beta y,\alpha z+\beta z)=(\alpha x,\alpha y,\alpha z)+(\beta x,\beta y,\beta z)$$

$$=\alpha(x,y,z)+\beta(x,y,z)=\alpha u+\beta u$$

b.3) $\underline{\alpha(u+v)=\alpha u+\alpha v}$

$$\alpha(u+v)=\alpha\left[(x,y,z)+(x',y',z')\right]=\alpha(x+x',y+y',z+z')$$

$$=(\alpha(x+x'),\alpha(y+y'),\alpha(z+z'))=(\alpha x+\alpha x',\alpha y+\alpha y',\alpha z+\alpha z')$$

$$=(\alpha x,\alpha y,\alpha z)+(\alpha x',\alpha y',\alpha z')=\alpha(x,y,z)+\alpha(x',y',z')=\alpha u+\alpha v$$

b.4) $\underline{1u=u.}$

$$1u=1(x,y,z)=(1.x,1.y,1.z)=(x,y,z)=u$$

de a) et b) on déduit que $\left(\mathbb{R}^3,+,.\right)$ ***est un*** $\mathbb{R}-$***espace vectoriel***

Exercice02 $v_1=(0,1,-2)$, $v_2=(1,1,0)$, $v_3=(-2,0,-2)$

1. $\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3=\lambda_1(0,1,-2)+\lambda_2(1,1,0),+\lambda_3(-2,0,-2)=(\lambda_2-2\lambda_3,\lambda_1+\lambda_2,-2\lambda_1-2\lambda_3)$

$$\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3=(0,0,0)\Leftrightarrow\begin{cases}\lambda_2-2\lambda_3=0....\times(-1)\\ \lambda_1+\lambda_2=0......\times(1)\\ -2\lambda_1-2\lambda_3=0\end{cases}\Leftrightarrow\begin{cases}2\lambda_3+\lambda_1=0......\times(1)\\ \lambda_1+\lambda_2=0\\ -2\lambda_1-2\lambda_3=0....\times(1)\end{cases}$$

$$\Leftrightarrow\begin{cases}-\lambda_1=0\\ \lambda_1+\lambda_2=0\\ -2\lambda_1-2\lambda_3=0\end{cases}\Leftrightarrow\begin{cases}\lambda_1=0\\ 0+\lambda_2=0\\ 0-2\lambda_3=0\end{cases}\Leftrightarrow\begin{cases}\lambda_1=0\\ \lambda_2=0\\ \lambda_3=0\end{cases}$$

Puisque $\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3=0_{\mathbb{R}^3}\Rightarrow\lambda_1=\lambda_2=\lambda_3=0$

Alors $v_1=(0,1,-2)$, $v_2=(1,1,0)$, $v_3=(-2,0,-2)$ ***sont linéairement indépendants*** $\Rightarrow$

$\Rightarrow\{v_1,v_2,v_3\}$ ***forme une Base de*** $\mathbb{R}^3$

2. $\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3=U\Leftrightarrow(\lambda_2-2\lambda_3,\lambda_1+\lambda_2,-2\lambda_1-2\lambda_3)=(t,5,-5)\Leftrightarrow$

$$\begin{cases}\lambda_2-2\lambda_3=t........\times(-1)\\ \lambda_1+\lambda_2=5..........\times(1)\\ -2\lambda_1-2\lambda_3=-5\end{cases}\Leftrightarrow\begin{cases}2\lambda_3+\lambda_1=-t+5......\times(1)\\ \lambda_1+\lambda_2=5\\ -2\lambda_1-2\lambda_3=-5.......\times(1)\end{cases}\Leftrightarrow\begin{cases}-2\lambda_1+\lambda_1=-t\\ \lambda_1+\lambda_2=5\\ -2\lambda_1-2\lambda_3=-5\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\lambda_1=t\\ t+\lambda_2=5\\ -2t-2\lambda_3=-5\end{cases}\Leftrightarrow\begin{cases}\lambda_1=t\\ \lambda_2=5-t\\ -2\lambda_3=-5+2t\end{cases}\Leftrightarrow\begin{cases}\lambda_1=t\\ \lambda_2=5-t\\ \lambda_3=\frac{5}{2}-t\end{cases}$$

En effet pour $t\in\mathbb{R}-\left\{0,5,\frac{5}{2}\right\}$ ***, on a :***

$$t(0,1,-2)+(5-t)(1,1,0)+\left(\tfrac{5}{2}-t\right)(-2,0,-2)=(0+5-t-5+2t,t+5-t+0,-2t+0-5+2t)$$
$$=(t,5,-5)$$

Exercice03

I $E=\left\{(x,y,z,t)\in\mathbb{R}^4;\ x-y-z+2t=0\right\}$

I.a) $0-0-0+2\times0=0\Rightarrow(0,0,0,0)\in E$

* $u_1=(x_1,y_1,z_1,t_1)\in E$ ***et*** $u_2=(x_2,y_2,z_2,t_2)\in E\Rightarrow x_1-y_1-z_1+2t_1=0$ ***et*** $x_2-y_2-z_2+2t_2=0$

$\Rightarrow(x_1-y_1-z_1+2t_1)+(x_2-y_2-z_2+2t_2)=0$

$\Rightarrow(x_1+x_2)-(y_1+y_2)-(z_1+z_2)+2(t_1+t_2)=0$

$\Rightarrow u_1+u_2=(x_1+x_2,y_1+y_2,z_1+z_2,t_1+t_2)\in E$

* $u=(x,y,z,t)\in E\Rightarrow x-y-z+2t=0\Rightarrow\lambda(x-y-z+2t)=\lambda x-\lambda y-\lambda z+2\lambda t=0$

$\Rightarrow\lambda u=(\lambda x,\lambda y,\lambda z,\lambda t)\in E$

On déduit donc que E est un $\mathbb{R}-$ ***espace vectoriel sur*** $\mathbb{R}^4$***.***

I.b) $x-y-z+2t=0\Rightarrow x=y+z-2t$

$$\Rightarrow(x,y,z,t)=(y+z-2t,y,z,t)=y(1,1,0,0)+z(1,0,1,0)+t(-2,0,0,1)$$

$\Rightarrow\{(1,1,0,0),(1,0,1,0),(-2,0,0,1)\}$ ***est une base pour l'espace E*** $\Rightarrow\dim(E)=3$

II. $F = \{(x, y, z, t) \in \mathbb{R}^4;\ x - y = 0,\ z = t = 0\}$

II.a $0 - 0 = 0, 0 = 0 = 0 \Rightarrow (0,0,0,0) \in F$

* $u_1 = (x_1, y_1, z_1, t_1) \in F$ *et* $u_2 = (x_2, y_2, z_2, t_2) \in F \Rightarrow$

$\Rightarrow x_1 - y_1 = 0, z_1 = t_1 = 0$ *et* $x_2 - y_2 = 0, z_2 = t_2 = 0$

$\Rightarrow (x_1 + x_2) - (y_1 + y_2) = x_1 - y_1 + x_2 - y_2 = 0, z_1 + z_2 = t_1 + t_2 = 0 \Rightarrow u_1 + u_2 \in F$

* $x - y = 0, z = t = 0 \Rightarrow \lambda x - \lambda y = \lambda (x - y) = 0$ *et* $\lambda z = \lambda t = 0 \Rightarrow$

$\Rightarrow \lambda u = \lambda (x, y, z) = (\lambda x, \lambda y, \lambda z) \in F$

On déduit donc que F ***est un*** $\mathbb{R}-$ ***espace vectoriel sur*** $\mathbb{R}^4$.

II.b $x - y = 0, z = t = 0 \Leftrightarrow x = y$ *et* $z = t = 0 \Rightarrow (x, y, z, t) = (x, x, 0, 0) = x(1,1,0,0) \Rightarrow \dim(F) = 1$

1. $E \cap F = \{u \in \mathbb{R}^4,$ ***tel que :*** $u = y(1,1,0,0) + z(1,0,1,0) + t(-2,0,0,1)$ *et* $u = x(1,1,0,0)\}$

$y(1,1,0,0) + z(1,0,1,0) + t(-2,0,0,1) = x(1,1,0,0) \Leftrightarrow x = y$ *et* $z = 0$ *et* $t = 0$

$\Rightarrow E \cap F = \{u \in \mathbb{R}^4,$ ***tel que :*** $u = x(1,1,0,0)\} = E$

$E + F = \{u \in \mathbb{R}^4,$ ***tel que :*** $u = y(1,1,0,0) + z(1,0,1,0) + t(-2,0,0,1) + x(1,1,0,0)\}$

$u = y(1,1,0,0) + z(1,0,1,0) + t(-2,0,0,1) + x(1,1,0,0) =$

$= (x + y)(1,1,0,0) + +z(1,0,1,0) + t(-2,0,0,1) = y'(1,1,0,0) + z(1,0,1,0) + t(-2,0,0,1) \in F$

$\Rightarrow E + F = F$

III.a $\dim(E \cap F) = \dim(E) = 1$ *et* $\dim(E + F) = \dim(F) = 3$

III.b $E + F = F \neq \mathbb{R}^4$ *et* $E \cap F = E \neq \{0\} \Rightarrow E \oplus F \neq \mathbb{R}^4$