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Solution de l'Exo 1

equation	*forme*	*type*
$xy' = (x-1)\,y$	$xy' - (x-1)\,y = 0$	*lineaire homogène*
$\left(1+y^2\right)y' = x$	$\left(1+y^2\right)dy = xdx$	*à variables separables*
$y'\sin x\cos x - 3y = -3y^{\frac{2}{3}}\sin^3 x$	$y' - \frac{3}{\sin x\cos x}y = -3\frac{\sin^2 x}{\cos x}y^{\frac{2}{3}}$	*Bernoulli*
$y' = \frac{x-y}{x+y}$	$y' = \frac{1-\frac{y}{x}}{1+\frac{y}{x}} = f\left(\frac{y}{x}\right)$	*homogène*
$y' - \frac{y}{1-x^2} = 1+x$	$y' - \frac{y}{1-x^2} - (1+x) = 0$	*lineaire complete*

Solution de l'Exo 2

1. $(1+\exp(x))\,yy' = \exp(x) \Leftrightarrow (1+\exp(x))\,y\frac{dy}{dx} = \exp(x) \Leftrightarrow (1+e^x)\,ydy = e^x x \Leftrightarrow ydy = \frac{1}{1+e^x}e^x dx \Leftrightarrow ydy = \frac{1}{1+t}dt$
   $\Leftrightarrow \int ydy = \int \frac{1}{1+t}dt \Leftrightarrow \frac{1}{2}y^2 = \ln|1+t| + c = \ln|1+e^x| + c = \ln(e^x+1) + c \Leftrightarrow y^2 = 2\left(\ln(e^x+1)+c\right)$
2. $\Leftrightarrow y = \pm\left[2\left(\ln(e^x+1)+c\right)\right]^{\frac{1}{2}}$ *et* $x \geq -c$

$$\boxed{y = \left[2\left(\ln(e^x+1)+c\right)\right]^{\frac{1}{2}} \text{ ou } y=-\left[2\left(\ln(e^x+1)+c\right)\right]^{\frac{1}{2}}}$$

3. $\tan(x)\sin^2(y)\,dx + \cos^2(x)\cos(y)\,dy = 0 \Leftrightarrow \frac{\tan(x)}{\cos^2(x)}dx = -\frac{\cos(y)}{\sin^2(y)}dy$
   $y'\frac{\cos y}{\sin^2(y)} + \frac{\tan(x)}{\cos^2(x)} = 0 \Leftrightarrow \frac{\sin x}{\cos^3(x)}dx = -\sin^{-2}(y)\,d(\sin(y)) \Leftrightarrow -\cos^{-3}(x)\,d(\cos(x)) = -\sin^{-2}(y)\,d(\sin(y))$
   $\Leftrightarrow \cos^{-3}(x)\,d(\cos(x)) = \sin^{-2}(y)\,d(\sin(y)) \Leftrightarrow \frac{\cos^{-2}(x)}{-2} + c = \frac{\sin^{-1}(y)}{-1} \Leftrightarrow -\frac{1}{\sin y} = -\frac{1}{2\cos^2 x} + c_1 = \frac{2c\cos^2 x - 1}{2\cos^2 x}$
   $\Leftrightarrow \sin y = \frac{2\cos^2 x}{-2c_1\cos^2 x+1} = \frac{2\cos^2 x}{c\cos^2 x+1}$,

$$\boxed{y = \arcsin\left(\frac{2\cos^2 x}{c\cos^2 x+1}\right) \text{ ou } y = \pi\text{-}\arcsin\left(\frac{2\cos^2 x}{c\cos^2 x+1}\right)}$$

   $\tan(x)\sin^2(y)\,dx + \cos^2(x)\cot(y)\,dy = 0 \Leftrightarrow \frac{\sin x}{\cos x}\sin^2 ydx + \cos^2 x\frac{\cos y}{\sin y}dy = 0$
   $\Leftrightarrow \frac{\sin x}{\cos^3 x}dx = -\frac{\cos y}{\sin^3 y}dy \Leftrightarrow -\cos^{-3}xd(\cos x) = -\sin^{-3}yd(\sin y)$
   $\Leftrightarrow \int\cos^{-3}xd(\cos x) = \int\sin^{-3}yd(\sin y) \Leftrightarrow c_1 - \frac{1}{2}\cos^{-2}x = -\frac{1}{2}\sin^{-2}y$
   $\Leftrightarrow \sin^{-2}y = -2c_1 + \cos^{-2}x = c + \frac{1}{\cos^2 x} = \frac{1+c\cos^2 x}{\cos^2 x} \Leftrightarrow \sin^2 y = \frac{\cos^2 x}{1+c\cos^2 x}$
   $\Leftrightarrow \sin y = \pm\frac{|\cos x|}{\sqrt{1+c\cos^2 x}}\tan(x)\sin^2(y)\,dx + \cos^2(x)\cos(y)\,dy = 0$

$$\boxed{y = \arcsin\left(\pm\frac{|\cos x|}{\sqrt{1+c\cos^2 x}}\right) \text{ ou } y = \pi - \arcsin\left(\pm\frac{|\cos x|}{\sqrt{1+c\cos^2 x}}\right)}$$

4. $\frac{\exp(y)}{\exp(y)+1}dy = \frac{1}{x}dx \Leftrightarrow \frac{1}{e^y+1}d(e^y) = \frac{1}{x}dx \Leftrightarrow \int\frac{1}{e^y+1}d(e^y) = \int\frac{1}{x}dx \Leftrightarrow \ln|e^y+1| = \ln|x| + c_1 = \ln\left(e^{c_1}|x|\right)$
   $\Leftrightarrow e^y + 1 = e^{c_1}|x| = |c_2||x| = |c_2 x| = \pm c_2 x = cx \Leftrightarrow e^y = cx - 1 \Leftrightarrow y = \ln(cx-1)$

$$\boxed{y = \ln(cx-1),\quad \text{avec } x > \frac{1}{c}}$$

5. $3\exp(x)\tan(y)\,dx + \frac{(1-\exp(x))}{\cos^2(y)}dy = 0 \Leftrightarrow 3\frac{1}{1-e^x}e^x dx = -\frac{1}{\cos^2(y)\tan(y)}dy$
   $\Leftrightarrow 3\frac{1}{1-e^x}d(e^x) = -\frac{1}{\tan(y)}d(\tan y) \Leftrightarrow 3\frac{1}{1-e^x}d(e^x) = -\frac{1}{\tan(y)}d(\tan y) \Leftrightarrow -3\ln(|e^x-1|) + c_1 = -\ln(|\tan y|)$
   $\Leftrightarrow |\tan y| = e^{-c_1}|e^x-1|^3 \Leftrightarrow \tan y = \pm e^{-c_1}(e^x-1) = c(e^x-1)^3$

$$\boxed{y = \arctan\left(c(e^x-1)^3\right)}$$

6. $y'\tan(x) = y \Leftrightarrow \frac{dy}{dx}\tan(x) = y \Leftrightarrow \frac{1}{y}dy = \frac{\cos x}{\sin x}dx = \frac{1}{\sin x}d(\sin x)$

$\Leftrightarrow \frac{1}{y}dy = \frac{1}{\sin x}d(\sin x) \Leftrightarrow \ln|y| = \ln|\sin x| + c_1 \Leftrightarrow |y| = e^{c_1}|\sin x| \Leftrightarrow y = \pm e^{c_1}\sin x = c\sin x$

$$\boxed{y = c\sin x}$$

7. $(x^2+1)\,y' = y^2+4 \Leftrightarrow (x^2+1)\frac{dy}{dx} = y^2+4 \Leftrightarrow \frac{dy}{y^2+4} = \frac{dx}{x^2+1}$

$\Leftrightarrow \int \frac{dx}{x^2+1} = \int \frac{dy}{y^2+4} = \frac{1}{4}\int \frac{2d\left(\frac{y}{2}\right)}{\left(\frac{y}{2}\right)^2+1} = \frac{1}{2}\arctan\left(\frac{y}{2}\right) \Leftrightarrow \arctan\left(\frac{y}{2}\right) = 2\int \frac{dx}{x^2+1} = c + 2\arctan x$

$\Leftrightarrow \frac{y}{2} = \tan(c + 2\arctan x) \Leftrightarrow y = 2\tan(c + 2\arctan x)$

$$\boxed{y = 2\tan(c + 2\arctan x)}$$

<u>Solution de l'Exo</u> 3

1. $y' = \frac{y}{x} - 1 = u - 1 \Rightarrow\ u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = u'x + u \Rightarrow u'x + u = u - 1 \Rightarrow u'x = -1 \Rightarrow \frac{du}{dx}x = -1 \Rightarrow du = -\frac{1}{x}dx$

$\Rightarrow \int du = -\int \frac{1}{x}dx \Rightarrow u = -\ln|x| + c \Rightarrow \frac{y}{x} = -\ln|x| + c \Rightarrow y = x(c - \ln|x|)$

$$\boxed{y = x(c - \ln|x|)}$$

2. $y' = -\frac{x+y}{x} \Leftrightarrow y' = -1 - \frac{y}{x} = -1 - u$

$u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = u'x + u \Rightarrow u'x + u = -1 - u \Rightarrow u'x = -1 - 2u \Rightarrow x\frac{du}{dx} = -1 - 2u$

$\Rightarrow \frac{du}{(1+2u)} = -\frac{dx}{x} \Rightarrow \frac{1}{2}\int \frac{2du}{(1+2u)} = -\int \frac{dx}{x} \Rightarrow \frac{1}{2}\ln|1+2u| = -\ln|x| + c_1 = \ln\left(\frac{c_1}{x}\right)$

$\Rightarrow \ln|1+2u| = 2\ln\left(\frac{c_1}{x}\right) = \ln\left(\frac{c_0^2}{x^2}\right) \Rightarrow |1+2u| = \frac{c_1^2}{x^2} \Rightarrow 1 + 2u = \pm\frac{c_1^2}{x^2} = \frac{c}{x^2} \Rightarrow u = \frac{1}{2}\left(\frac{c}{x^2} - 1\right)$

$\Rightarrow \frac{y}{x} = \frac{1}{2}\left(\frac{c}{x^2} - 1\right) \Rightarrow y = \frac{1}{2}\left(\frac{c}{x} - x\right)$

$$\boxed{y = \frac{1}{2}\left(\frac{c}{x} - x\right)}$$

3. $(x-y)\,ydx - x^2dy = 0 \Leftrightarrow \frac{dy}{dx} = \frac{xy-y^2}{x^2} = \frac{y}{x} - \frac{y^2}{x^2} = u - u^2$

$y = ux \Rightarrow y' = xu' + u = u - u^2 \Rightarrow xu' = -u^2 \Rightarrow \frac{du}{dx} = -\frac{u^2}{x} \Rightarrow -\frac{1}{u^2}du = \frac{1}{x}dx \Rightarrow -\int \frac{1}{u^2}du = \int \frac{1}{x}dx$

$\Rightarrow \frac{1}{u} = \ln|x| + c \Rightarrow \frac{y}{x} = u = \frac{1}{c+\ln|x|} \Rightarrow y = \frac{x}{c+\ln|x|}$

$$\boxed{y = \frac{x}{c+\ln|x|}}$$

4. $(x^2+y^2)\,dx - 2xydy = 0 \Leftrightarrow \frac{dy}{dx} = \frac{x^2+y^2}{2xy} = \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) = \frac{1}{2}\left(\frac{1}{u} + u\right)$

$\frac{y}{x} = u \Leftrightarrow y = ux \Rightarrow y' = u'x + u = \frac{1}{2}\left(\frac{1}{u} + u\right) \Rightarrow u'x = \frac{1}{2u} - \frac{1}{2}u = \frac{1}{2}\left(\frac{1}{u} - u\right) = \frac{1-u^2}{2u}$

$\Rightarrow x\frac{du}{dx} = \frac{1-u^2}{2u} \Rightarrow \frac{2u}{1-u^2}du = \frac{1}{x}dx \Rightarrow -\int \frac{2u}{1-u^2}du = \int \frac{1}{x}dx$

$\Rightarrow \ln\left|1-u^2\right| = \ln|x| + c_1 = -\ln\left(e^{c_1}|x|\right) = \ln\left(e^{-c_1}|x|^{-1}\right) \Rightarrow 1 - u^2 = \pm\frac{c_2}{x} = \frac{c}{x} \Rightarrow u^2 - 1 = \frac{c}{x}$

$\Rightarrow u = \pm\left(1 + \frac{c}{x}\right)^{\frac{1}{2}} \Rightarrow \frac{y}{x} = \pm\left(1 + \frac{c}{x}\right)^{\frac{1}{2}} \Rightarrow y = \pm x\left(1 + \frac{c}{x}\right)^{\frac{1}{2}} = \pm x\sqrt{1 + \frac{c}{x}} = \pm\sqrt{\left(x^2 1 + \frac{c}{x}\right)} = \pm\sqrt{x(c+x)}$

$$\boxed{y = \sqrt{x(c+x)} \quad ou \quad y = -\sqrt{x(c+x)}}$$

5. $xdy - ydx = \sqrt{x^2+y^2} \Leftrightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{1}{x}\sqrt{x^2+y^2} = \frac{y}{x} + \sqrt{\frac{1}{x^2}(x^2+y^2)} = \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = u + \sqrt{1+u^2}$

$\frac{y}{x} = u \Leftrightarrow y = ux \Rightarrow \frac{dy}{dx} = y' = u'x + u = u + \sqrt{1+u^2} \Rightarrow u'x = \sqrt{1+u^2} \Rightarrow x\frac{du}{dx} = \sqrt{1+u^2}$

$\Rightarrow \frac{1}{\sqrt{1+u^2}}du = \frac{1}{x}dx \Rightarrow \int \frac{1}{\sqrt{1+u^2}}du = \int \frac{1}{x}dx \Rightarrow arcsinh\,(u) = c_1 + \ln|x|$

$\Rightarrow u = \sinh(c_1 + \ln|x|) = \frac{e^{c_1+\ln|x|} - e^{-(c_1+\ln|x|)}}{2} = \frac{|c_2||x| - |c_2|^{-1}|x|^{-1}}{2} = \frac{(\pm c_2 x)^2 - 1}{\pm 2c_2 x} = \frac{c^2x^2-1}{2cx}$

$\Rightarrow \frac{y}{x} = \frac{c^2x^2-1}{2cx} \Rightarrow y = \frac{1}{2}\left(cx^2 - \frac{1}{c}\right)$

$$\boxed{y = \frac{1}{2}\left(cx^2 - \frac{1}{c}\right)}$$

Solution de l'Exo 4

1. $y' - \frac{y}{x} = x$

Cette equation est lineaire non homogène

Soit l'équation homogène :

$y_h' - \frac{y_h}{x} = 0 \Leftrightarrow \frac{dy_h}{dx} = \frac{y_h}{x} \Leftrightarrow \frac{dy_h}{y_h} = \frac{dx}{x} \Leftrightarrow \int \frac{dy}{y_h} = \int \frac{dx}{x} \Leftrightarrow \ln|y_h| = c_1 + \ln|x| = \ln(e^{c_1}|x|)$

$\Leftrightarrow |y_h| = e^{c_1}|x| \Leftrightarrow y_h = \pm e^{c_1}x = cx$

Par la variation de la constante on cherche $y_p = xc(x)$*verifiant l'équation homogène :* $y_p' - \frac{y_p}{x} = x$

$y_p' - \frac{y_p}{x} = x \Leftrightarrow (xc(x))' - \frac{xc(x)}{x} = xc'(x) + c(x) - c(x) = x \Leftrightarrow xc'(x) = x \Leftrightarrow c'(x) = 1$

$\Leftrightarrow c(x) = x + a \Rightarrow y_p = xc(x) = x(x+k) = x^2 + ax \Rightarrow y_g = y_p + y_h = x^2 + ax + cx = x^2 + (a+c)x = x^2 + kx$

$$\boxed{y = x^2 + kx}$$

2. $\left(1+y^2\right)dx = \left(\sqrt{1+y^2}\sin y - xy\right)dy \Leftrightarrow P(x,y)\,dx + Q(x,y)\,dy = 0$

tels que : $P(x,y) = 1 + y^2$ *et* $Q(x,y) = -\sqrt{1+y^2}\sin y + xy$

$\frac{\partial P}{\partial y} = 2y$ *et* $\frac{\partial Q}{\partial x} = y \Rightarrow \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ $\Rightarrow$*cette equation est non exacte,*

son facteur integrant est $F(y) = e^{\int \frac{1}{p}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dy} = e^{\int \frac{1}{1+y^2}(y-2y)dy} = \frac{1}{\sqrt{y^2+1}} = \left(y^2+1\right)^{-\frac{1}{2}}$

$\Rightarrow \left(y^2+1\right)^{-\frac{1}{2}}\left(1+y^2\right)dx - \left(y^2+1\right)^{-\frac{1}{2}}\left(\sqrt{1+y^2}\sin y - xy\right)dy = 0$

$\Rightarrow \left(1+y^2\right)^{\frac{1}{2}}dx + \left(-\sin y + x\frac{y}{(y^2+1)^{\frac{1}{2}}}\right)dy = 0$

$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial y}\left(1+y^2\right)^{\frac{1}{2}} = \frac{1}{2}\frac{2y}{(1+y^2)^{\frac{1}{2}}} = \frac{y}{(1+y^2)^{\frac{1}{2}}}$

$\frac{\partial}{\partial y}N(x,y) = \frac{\partial}{\partial x}\left(-\sin y + x\frac{y}{(y^2+1)^{\frac{1}{2}}}\right) = \frac{y}{(y^2+1)^{\frac{1}{2}}}$

$\Rightarrow u(x,y) = \int M(x,y)\,dx + k(y) = \int \left(1+y^2\right)^{\frac{1}{2}}dx + k(y) = \left(1+y^2\right)^{\frac{1}{2}}x + k(y)$

$N(x,y) = \frac{\partial u(x,y)}{\partial y} = \frac{1}{2}x\frac{2y}{(1+y^2)^{\frac{1}{2}}} + k'(y) = -\sin y + x\frac{y}{(y^2+1)^{\frac{1}{2}}} \Rightarrow k'(y) = -\sin y$

$\Rightarrow k(y) = -\int \sin y dy = \cos y + c \Rightarrow u(x,y) = \left(1+y^2\right)^{\frac{1}{2}}x + \cos y + c$

d'autre part on a aussi :

$u(x,y) = \int N(x,y)\,dy + l(x) = \int \left(-\sin y + x\frac{y}{(y^2+1)^{\frac{1}{2}}}\right)dy + l(x) = \cos y + x\sqrt{y^2+1} + l(x)\,.$

$M(x,y) = \frac{\partial u(x,y)}{\partial x} = \sqrt{y^2+1} + l'(x) = \left(1+y^2\right)^{\frac{1}{2}} \Rightarrow l'(x) = 0 \Rightarrow l(x) = cste$

$$\boxed{\left(1+y^2\right)^{\frac{1}{2}}x + \cos y = cste}$$

3. $y^2dx-(2xy+3)\,dy=0 \Leftrightarrow P(x,y)\,dx+Q(x,y)\,dy=0$

$P(x,y)=y^2$ *et* $Q(x,y)=-(2xy+3)$

$\frac{\partial}{\partial y}(P(x,y))=2y$ *et* $\frac{\partial}{\partial x}(Q(x,y))=-2y \Rightarrow \frac{\partial}{\partial y}(P(x,y))\neq\frac{\partial}{\partial x}(Q(x,y))$

$\Rightarrow$*cette equation est non exacte, son facteur integrant est*

$F(y)=e^{\int\frac{1}{p}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dy}=e^{\int\frac{1}{y^2}(-2y-2y)dy}=\frac{1}{y^4}$

$\Rightarrow \frac{1}{y^4}\left(y^2dx-(2xy+3)\,dy\right)=0 \Rightarrow \frac{1}{y^2}dx-\frac{2xy+3}{y^4}dy=0 \Rightarrow M(x,y)\,dx+N(x,y)\,dy=0$

$\frac{\partial}{\partial y}(M(x,y))=-2y^{-3}$ *et* $\frac{\partial}{\partial x}(N(x,y))=\frac{\partial}{\partial x}\left(-\frac{2xy+3}{y^4}\right)=-\frac{2y}{y^4}=-2y^{-3}$

$\Rightarrow u(x,y)=\int M(x,y)\,dx+k(y)=\int\frac{1}{y^2}dx+k(y)=\frac{x}{y^2}+k(y)$

$N(x,y)=\frac{\partial u(x,y)}{\partial y}=\frac{\partial\left(\frac{x}{y^2}+k(y)\right)}{\partial y}=-2xy^{-3}+k'(y)=-\frac{2xy+3}{y^4}\Rightarrow k'(y)=-\frac{3}{y^4}=-3y^{-4}$

$\Rightarrow k(y)=y^{-3}+c \Rightarrow u(x,y)=\frac{x}{y^2}+k(y)=\frac{x}{y^2}+y^{-3}+c$

d'autre part on a aussi :

$u(x,y)=\int N(x,y)\,dy+l(x)=\int N(x,y)\,dy+l(x)=-\int\frac{2xy+3}{y^4}dy+l(x)=\frac{1}{y^3}(xy+1)+l(x)$

$M(x,y)=\frac{\partial u(x,y)}{\partial x}=\frac{1}{y^3}y+l'(x)=\frac{1}{y^2}\Rightarrow l'(x)=0\Rightarrow l(x)=cste$

$\Rightarrow u(x,y)=\frac{1}{y^3}(xy+1)+l(x))=\frac{x}{y^2}+y^{-3}+cste$

$$\boxed{\frac{x}{y^2}+y^{-3}=cste}$$

Solution de l'Exo 5

1. $xy'+y-e^x=0;\quad y(a)=b,$

Soit l'équation hiomogène : $xy_h'+y_h=0\Leftrightarrow\frac{dy_h}{y_h}=-\frac{dx}{x}\Leftrightarrow\int\frac{dy_h}{y_h}=-\int\frac{dx}{x}\Leftrightarrow\ln|y_h|=-\ln|x|+c_1$

$\Leftrightarrow\ln|y_h|=\ln\left(e^{c_1}|x|^{-1}\right)\Leftrightarrow y_h=\pm e^{c_1}x^{-1}=cx^{-1}=\frac{k}{x}\Leftrightarrow y_h=\frac{c_2}{x}$

Soit $y_p=\frac{k(x)}{x}$*, vérifiant l'équation non homogène :*

$xy_p'+y_p-e^x=0\Leftrightarrow xy_p'+y_p-e^x=x\left(\frac{xk'-k}{x^2}\right)+\frac{k}{x}-e^x=k'-e^x=0\Rightarrow k'(x)=e^x\Rightarrow k(x)=e^x+c_3$

$\Rightarrow y_p=\frac{k(x)}{x}=\frac{e^x+c_3}{x}=\frac{e^x}{x}+\frac{c_3}{x}\Rightarrow y_g=y_p+y_h=\frac{e^x}{x}+\frac{c_2}{x}+\frac{c_3}{x}=\frac{e^x}{x}+\frac{c_2+c_3}{x}=\frac{e^x}{x}+\frac{c}{x}$

$y_p(a)=b\Leftrightarrow\frac{e^a}{a}+\frac{c}{a}=b\Leftrightarrow\frac{c}{a}=b-\frac{e^a}{a}\Leftrightarrow c=ab-e^a$

$$\boxed{y=\frac{e^x}{x}+\frac{ab-e^a}{x}}$$

2. $y'-\frac{y}{1-x^2}-1-x=0;,\left\{\frac{\int\sqrt{x-1}\sqrt{x+1}\,dx}{\sqrt{x-1}}\sqrt{x+1}+\frac{C_{19}}{\sqrt{x-1}}\sqrt{x+1}\right\},\quad y(0)=0,$

Soit l'équation homogène : $y_h'-\frac{y_h}{1-x^2}=0\Leftrightarrow y_h'=\frac{y_h}{1-x^2}\Leftrightarrow\frac{dy_h}{dx}=\frac{y_h}{1-x^2},$

$\Leftrightarrow\frac{dy_h}{y_h}=\frac{dx}{1-x^2}\Leftrightarrow\int\frac{dy_h}{y_h}=\int\frac{dx}{1-x^2}\Leftrightarrow\ln|y_h|=\frac{1}{2}\ln(|x+1|)-\frac{1}{2}\ln|x-1|=\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+c_1=\ln\left(e^{c_1}\left|\frac{x+1}{x-1}\right|^{\frac{1}{2}}\right)$

$\Leftrightarrow|y_h|=e^{c_1}\left|\frac{x+1}{x-1}\right|^{\frac{1}{2}}\Leftrightarrow y_h=\pm e^{c_1}\left|\frac{x+1}{x-1}\right|^{\frac{1}{2}}=c_2\left|\frac{x+1}{x-1}\right|^{\frac{1}{2}}$

puisque $y(0)=0$ *alors* $y_h=c_2\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}},\ x\in[-1,1]$

Soit $y_p=k(x)\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$*, vérifiant l'équation non homogène :* $y_p'-\frac{y_p}{1-x^2}-1-x=0.$

$\Rightarrow y_p'=k'(x)\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}+k(x)\frac{1}{2}\frac{1-x-(-1)(1+x)}{(1-x)^2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}-1}=k'(x)\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}+k(x)\frac{1}{(x-1)^2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}$

$\Rightarrow k'(x)\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} + k(x)\ \frac{1}{(x-1)^2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}} - \frac{1}{1-x^2}k(x)\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} - 1 - x = 0$:

$\Rightarrow k'(x)\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} - 1 - x = 0 \Rightarrow k'(x) = (1+x)\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}} = (1+x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$

$\Rightarrow k(x) = \int \sqrt{1-x^2}dx = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2}$

$y_p = \left(\frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2}\right)\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} \Rightarrow y_g = y_h + y_p = c\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} + \left(\frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2}\right)\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$

$y_g(0) = 0 \Leftrightarrow c + 0 = 0 \Rightarrow c = 0$

$$\boxed{y(x) = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right)\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}}$$

3. $y' - y\tan x = \frac{1}{\cos x}$;, $y(0) = 0$

Soit l'équation homogène $y_h' - y_h\tan x = 0 \Leftrightarrow \frac{dy_h}{dx} - y_h\tan x = 0 \Leftrightarrow \frac{dy_h}{y_h} = \frac{dx}{\tan x} = \frac{\sin x}{\cos x}dx = -\frac{d(\cos x)}{\cos x}$

$\Leftrightarrow \ln|y_h| = -\int \frac{d(\cos x)}{\cos x} = c_1 - \ln|\cos x| \Leftrightarrow |y_h| = e^{c_1}|\cos x|^{-1} \Leftrightarrow y_h = \frac{\pm e^{c_1}}{\cos x} = \frac{c_2}{\cos x}$

Soit l'équation $y_p = \frac{k(x)}{\cos x}$ *la solution de l'equation non homogène* $y_p' - y_p\tan x = \frac{1}{\cos x}$

$\Rightarrow \frac{k'\cos x + k\sin x}{\cos^2 x} - \frac{k}{\cos x}\tan x = \frac{1}{\cos x} \Rightarrow \frac{k'}{\cos x} = \frac{1}{\cos x} \Rightarrow \frac{dk}{dx} = 1 \Rightarrow dk = dx \Rightarrow k = x + c \Rightarrow y_p = \frac{k(x)}{\cos x} = \frac{x+c}{\cos x}$

$y_p(0) = 0 \Rightarrow c = 0$

$$\boxed{y = \frac{x}{\cos x}}$$

<u>Solution de l'Exo</u> 6

1. $y' + \frac{y}{x} = -xy^2$,

Equation de Bernoullt $\Rightarrow \frac{1}{y^2}y' + \frac{1}{y^2}\frac{y}{x} = -x \Rightarrow \frac{1}{y^2}y' + \frac{1}{y}\frac{1}{x} = -x$

Soit $u = \frac{1}{y} \Rightarrow u' = -\frac{1}{y^2}y' \Rightarrow -u' + \frac{1}{x}u = -x$,

Soit l'equation homogène : $-u_h' + \frac{1}{x}u_h = 0 \Leftrightarrow -\frac{du_h}{dx} + \frac{1}{x}u_h = 0 \Leftrightarrow \frac{du_h}{dx} = \frac{u_h}{x} \Leftrightarrow \frac{du_h}{u_h} = \frac{dx}{x}$

$\Leftrightarrow \int \frac{du_h}{u_h} = \int \frac{dx}{x} \Leftrightarrow \ln|u_h| = c_1 + \ln|x| = \ln\left(e^{c_1}|x|\right) \Leftrightarrow |u_h| = e^{c_1}|x| \Leftrightarrow u_h = \pm e^{c_1}|x| = c_2 x$

Soit $u_p = xk(x)$ *solution particuilere de l'equation non homogène* $-u_p' + \frac{u_p}{x} = -x$

$\Rightarrow -(xk' + k) + x\frac{k}{x} = -x \Rightarrow -xk' = -x \Rightarrow k' = 1 \Rightarrow \frac{dk}{dx} = 1 \Rightarrow dk = dx \Rightarrow k = x + c_3$

$\Rightarrow u_p = xk(x) = x(x + c_3) \Rightarrow u = u_p + u_h = x(x + c_3) + c_2 x = x^2 + cx$

$u = \frac{1}{y} \Rightarrow y = \frac{1}{u} = \frac{1}{x^2+cx}$

$$\boxed{y = \frac{1}{x^2+cx}}$$

2. $2xyy' - y^2 + x = 0$,

Soit $u = y^2 \Rightarrow u' = 2yy' \Rightarrow xu' - u + x = 0$,

Soit l''equation homogène : $xu_h' - u_h = 0 \Rightarrow \frac{xdu_h}{dx} - u_h = 0 \Rightarrow \frac{du_h}{u_h} = \frac{1}{x}dx$,

$\Rightarrow \ln|u_h| = c_1 + \ln|x| \Rightarrow |u_h| = e^{c_1}|x| \Rightarrow u_h = \pm e^{c_1}x = c_2 x$

Soit $u_p = xk(x)$ *une solution particuliere de* $xu_p' - u_p + x = 0 \Rightarrow x(k + xk') - xk + x = 0 \Rightarrow x^2k' = -x$

$\Rightarrow \frac{dk}{dx} = \frac{-1}{x} \Rightarrow k = -\int \frac{dx}{x} = -\ln|x| + c_3 \Rightarrow u_p = xk(x) = x(-\ln|x| + c_3)$

$\Rightarrow u = u_p + u_h = x(-\ln|x| + c_3) + c_2 x = -x\ln x + cx \Rightarrow y^2 = -x\ln x + cx \Rightarrow$

$$\boxed{y = \sqrt{cx - x\ln x} \quad ou \quad y = -\sqrt{cx - x\ln x}}$$

3. $y' - y = 2\sqrt{y}e^{-x} \Leftrightarrow y'y^{-\frac{1}{2}} - y^{\frac{1}{2}} = 2e^{-x}$,

   Soit $u = y^{\frac{1}{2}} \Rightarrow u' = \frac{1}{2}y'y^{-\frac{1}{2}} \Rightarrow 2u' - u = 2e^{-x}$

   Soit l'équation homogène $2u'_h - u_h = 0 \Leftrightarrow \frac{du_h}{dx} = \frac{1}{2}u_h \Leftrightarrow \frac{du_h}{u_h} = \frac{1}{2}dx$

   $\Leftrightarrow \int \frac{du_h}{u_h} = \frac{1}{2}\int dx \Leftrightarrow \ln|u_h| = \frac{1}{2}x + c_1 \Leftrightarrow |u_h| = e^{c_1}e^{\frac{1}{2}x} \Leftrightarrow u_h = \pm e^{c_1}e^{\frac{1}{2}x} \Leftrightarrow u_h = c_2 e^{\frac{1}{2}x}$

4. *Soit* $u_p = e^{\frac{1}{2}x}k(x)$ *la solution de l'equation non homogène :* $u'_p - u_p = 2e^{-x}$

   $\Leftrightarrow 2\left(\frac{1}{2}ke^{\frac{1}{2}x} + k'e^{\frac{1}{2}x}\right) - e^{\frac{1}{2}x}k(x) = 2e^{-x} \Leftrightarrow 2k'e^{\frac{1}{2}x} = 2e^{-x} \Leftrightarrow k'e^{\frac{1}{2}x} = e^{-x}$

   $\Leftrightarrow k'e^{\frac{1}{2}x} = e^{-x} \Leftrightarrow k' = e^{-x}e^{-\frac{1}{2}x} = e^{-\frac{3}{2}x} \Leftrightarrow \frac{dk}{dx} = e^{-\frac{3}{2}x} \Leftrightarrow k = \int e^{-\frac{3}{2}x}dx = \frac{1}{-\frac{3}{2}}e^{-\frac{3}{2}x} = -\frac{2}{3}e^{-\frac{3}{2}x} + c_3$

   $u_p = e^{\frac{1}{2}x}k(x) = u_p = e^{\frac{1}{2}x}\left(-\frac{2}{3}e^{-\frac{3}{2}x} + c_3\right) = -\frac{2}{3}e^{-x} + c_3 e^{\frac{1}{2}x}$

   $\Rightarrow u = u_p + u_h = -\frac{2}{3}e^{-x} + c_3 e^{\frac{1}{2}x} + c_2 e^{\frac{1}{2}x} = -\frac{2}{3}e^{-x} + ce^{\frac{1}{2}x}$

   $y^{\frac{1}{2}} = u \Leftrightarrow y = u^2 = \left(-\frac{2}{3}e^{-x} + ce^{\frac{1}{2}x}\right)^2$

$$\boxed{y = \frac{1}{9e^{2x}}\left(3ce^{\frac{3}{2}x} - 2\right)^2}$$

<u>Solution de l'Exo</u> 7

$$\left(x^3 - 1\right)y' - y^2 - x^2 y + 2x = 0 \qquad (1)$$

1. $y_0 = x^2$ *est une solution particulière de 1, car*

$$\begin{aligned}\left(x^3 - 1\right)y'_0 - y_0^2 - x^2 y_0 + 2x &= \left(x^3 - 1\right)2x - x^4 - x^2x^2 + 2x \\ &= 2x^4 - 2x - 2x^4 + 2x = 0\end{aligned}$$

2. *Soit* $y = y_0 + z = x^2 + z \Rightarrow y' = 2x + z' \Rightarrow$

   $\left(x^3 - 1\right)y' - y^2 - x^2 y + 2x = \left(x^3 - 1\right)(2x + z') - \left(x^2 + z\right)^2 - x^2\left(x^2 + z\right) + 2x = z'x^3 - 3x^2 z - z^2 - z' = 0 \Rightarrow$

   $\left(x^3 - 1\right)z' - 3x^2 z - z^2 = 0, \Rightarrow z^{-2}\left(x^3 - 1\right)z' - 3x^2 z^{-1} - 1 = 0$

   Soit $u = z^{-1} \Rightarrow u' = -z^{-2}z' \Rightarrow -\left(x^3 - 1\right)u' - 3x^2 u - 1 = 0,$

   Soit l'équation homogène : $-\left(x^3 - 1\right)u'_h - 3x^2 u_h = 0 \Leftrightarrow \frac{du_h}{dx} = -\frac{3x^2 u_h}{x^3 - 1}$

   $\Leftrightarrow \frac{du_h}{u_h} = \frac{3x^2}{x^3 - 1}dx \Leftrightarrow \int \frac{du_h}{u_h} = -\int \frac{3x^2}{x^3 - 1}dx$

   $\Leftrightarrow \ln|u_h| = \ln\left|x^3 - 1\right|^{-1} + c_1 = \ln\left(e^{c_1}\left|x^3 - 1\right|^{-1}\right) \Leftrightarrow |u_h| = e^{c_1}\left|x^3 - 1\right|^{-1}$

   $\Leftrightarrow u_h = \pm e^{c_1}\left(x^3 - 1\right)^{-1} = c_2\left(x^3 - 1\right)^{-1}$

   Soit $u_p = \left(x^3 - 1\right)^{-1}k(x)$ *la solution particuliere vérifiant* $\left(x^3 - 1\right)u'_p - 3x^2 u_p - 1 = 0,$

   $\Leftrightarrow -\left(x^3 - 1\right)\left[-3kx^2\left(x^3 - 1\right)^{-2} + k'\left(x^3 - 1\right)^{-1}\right] - 3x^2\left(x^3 - 1\right)^{-1}k(x) - 1 = 0$

   $\Leftrightarrow -k' - 1 = 0 \Leftrightarrow \frac{dk}{dx} = -1 \Leftrightarrow k = \int dx = c_3 - x \Rightarrow u_p = \left(x^3 - 1\right)^{-1}k(x) = \left(x^3 - 1\right)^{-1}(c_3 - x)$

   $\Rightarrow u = u_p + u_h = \left(x^3 - 1\right)^{-1}(c_3 - x) + c_2\left(x^3 - 1\right)^{-1} = \left(x^3 - 1\right)^{-1}(c - x)$

   $\Rightarrow z^{-1} = \left(x^3 - 1\right)^{-1}(c - x) \Rightarrow z = \frac{x^3 - 1}{c - x} \Rightarrow y = y_0 + z = x^2 + \frac{x^3 - 1}{c - x}$

$$\boxed{y = x^2 + \frac{x^3 - 1}{c - x}}$$

Solution de l'Exo 8

$y'' - 5y' + 6y = 0$

cette equation differentielle lineaire du deuxieme ordre admet des solutions de la forme $y = re^x$

$y = re^x \Rightarrow y' = r^2e^x \Rightarrow y'' = r^3e^x \Rightarrow r^3e^x - 5r^2e^x + 6re^x = 0 \Rightarrow r^2 - 5r + 6 = 0 \Rightarrow r \in \{1, 5\} \Rightarrow$

$$\boxed{y = c_1e^x + c_2e^{5x}}$$

$y'' + 2y' + y = 0 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r_1 = r_2 = -1$

$$\boxed{y = (ax + b)\, e^{-x}}$$

$y'' - 9y = 0 \Rightarrow r^2 - 9 = 0 \Leftrightarrow r \in \{3, -3\}$

$$\boxed{y = ae^{3x} + be^{-3x}}$$

Solution de l'Exo 9

1. $y'' + y' - 2y = e^{-x} \ \Rightarrow r^2 + r - 2 = 0 \Leftrightarrow (r+2)(r-1) = 0 \Leftrightarrow r \in \{-2, 1\} \Rightarrow y_h = ae^x + be^{-2x}$

   $r \neq -1 \Rightarrow y_p = ce^{-x} \Rightarrow y_p' = -ce^{-x} \Rightarrow y_p'' = ce^{-x} \Rightarrow ce^{-x} - ce^{-x} - 2ce^{-x} = e^{-x}$

   $\Rightarrow -2c = 1 \Rightarrow c = -\frac{1}{2} \Rightarrow y_p = -\frac{1}{2}e^{-x}$

   $$\boxed{y = ae^x + be^{-2x} - \tfrac{1}{2}e^{-x}}$$

2. $y'' - y' - 2y = x^2 - 1 = \left(x^2 - 1\right)e^{0x}, \Rightarrow r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0 \Rightarrow r \in \{-2, 1\} \Rightarrow y_h = c_1e^{-x} + c_2e^{2x}$

   $r \neq 0 \Rightarrow y_p = x^0\left(ax^2 + bx + c\right)e^{0x} = \left(ax^2 + bx + c\right) \Rightarrow y_p' = 2ax + b \Rightarrow y_p'' = 2a$

   $\Rightarrow 2a - (2ax + b) - 2\left(ax^2 + bx + c\right) = -2ax^2 - 2(a+b)x + 2a - b - 2c = x^2 - 1$

   $\Rightarrow -2a = 1$ *et* $-2(a+b) = 0$ *et* $2a - b - 2c = -1,$

   $\Rightarrow a = -\frac{1}{2}$ *et* $b = \frac{1}{2}$ *et* $c = -\frac{1}{4} \Rightarrow y_p = ax^2 + bx + c = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4}$

   $\Rightarrow y = y_h + y_p = c_1e^{-x} + c_2e^{2x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{4}$

   $$\boxed{y = c_1e^{-x} + c_2e^{2x} - \tfrac{1}{2}x^2 + \tfrac{1}{2}x - \tfrac{1}{4}}$$

3. $y'' + y' - 2y = e^x \Rightarrow r^2 + r - 2 = 0 \Leftrightarrow (r+2)(r-1) = 0 \Rightarrow r \in \{-2, 1\} \Rightarrow y_h = c_1e^{-2x} + c_2e^x$

   $r = 1 \Rightarrow y_p = (ax + b)\,e^x \Rightarrow y_p' = ae^x + (ax + b)\,e^x = (ax + a + b)\,e^x$

   $\Rightarrow y_p'' = ae^x + (ax + a + b)\,e^x = (ax + 2a + b)\,e^x$

   $\Rightarrow y'' + y' - 2y = (ax + 2a + b)\,e^x + (ax + a + b)\,e^x - 2(ax + b)\,e^x = 3ae^x$

   $\Rightarrow 3ae^x = e^x \Rightarrow 3a = 1 \Rightarrow a = \frac{1}{3} \Rightarrow y_p = \left(\frac{1}{3}x + b\right)e^x$

   $\Rightarrow y = y_h + y_p = c_1e^{-2x} + c_2e^x + \left(\frac{1}{3}x + b\right)e^x = c_1e^{-2x} + \left(\frac{1}{3}x + c_3\right)e^x$

   $$\boxed{y = k_1e^{-2x} + \left(\tfrac{1}{3}x + k_2\right)e^x}$$

Solution de l'Exo 10

1. $y'' + y = \tan x,$

Premiere methode :

$$r^2 + 1 = 0 \Rightarrow r \in \{-i, i\} \Rightarrow y_1 = \cos x \text{ et } y_2 = \sin x \text{ et } f(x) = \tan x$$

$$\Rightarrow y_h = c_1 \cos x + c_2 \sin x \Rightarrow y_p = u_1(x)\cos + u_2(x)\sin x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ \frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & \frac{\partial y_2}{\partial x} \end{vmatrix} = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = -\frac{\sin^2 x}{\cos x}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ \frac{\partial y_1}{\partial x} & f(x) \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \sin x$$

$$\Rightarrow u_1' = -\frac{\sin^2 x}{\cos x} \Rightarrow u_1 = -\int \frac{\sin^2 x}{\cos^2 x}\cos x = -\int \frac{\sin^2 x}{1-\sin^2 x} d(\sin x) = \int \frac{-t^2}{1-t^2} dt$$

$$\frac{-t^2}{1-t^2} = \frac{1-t^2-1}{1-t^2} = 1 - \frac{1}{1-t^2} = 1 - \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right)$$

$$\Rightarrow \int \frac{-t^2}{1-t^2} dt = t - \frac{1}{2}\left((\ln(1+t) - \ln(1-t))\right) = t - \frac{1}{2}\left(\ln\left(\frac{1+t}{1-t}\right)\right)$$

$$\Rightarrow u_1 = \sin x - \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) \Rightarrow u_2' = \frac{\sin x}{1} = \sin x \Rightarrow u_2(x) = \int \sin x dx = -\cos x$$

$$\Rightarrow y_p = u_1 \cos + u_2 \sin x = \left(\sin x - \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x}\right)\cos x + (-\cos x)\sin x = -\frac{\cos x}{2}\ln\frac{1+\sin x}{1-\sin x}$$

$$\Rightarrow y = y_h + y_p = c_1 \cos x + c_2 \sin x - \frac{\cos x}{2}\ln\frac{1+\sin x}{1-\sin x}$$

Deuxième methode :

$$y_p = k_1(x)\cos(x) \Rightarrow y_p' = k_1' \cos(x) - k_1 \sin x$$

$$\Rightarrow y'' = k_1'' \cos(x) - k_1' \sin x - k_1' \sin x - k_1 \cos x$$

$$\Rightarrow y_p'' + y_p = k_1'' \cos(x) - 2k_1' \sin x = \tan x$$

Soit $u(x) = k_1'(x) \Rightarrow u' \cos(x) - 2u \sin x = \tan x,$

$$u_h' \cos(x) - 2u_h \sin x = 0 \Rightarrow \frac{u_h'}{u_h} = 2\frac{\sin x}{\cos x} = -2\frac{(\cos x)'}{\cos x}$$

$$\Rightarrow \ln|u_h| = -2\ln|\cos x| + a = \ln|\cos x|^{-2} + a$$

$$\Rightarrow |u_h| = e^a |\cos x|^{-2} \Rightarrow u_h = \pm e^a (\cos x)^2 = b\cos^{-2} x = \frac{b}{\cos^2 x}$$

Soit $u_p = \frac{k_2(x)}{\cos^2 x}$, *solution particuliere de* $u_p' \cos(x) - 2u_p \sin x = \tan x$

$$\Rightarrow u_p' = \left(k_2 \cos^{-2} x\right)' = k_2' \cos^{-2} x + 2k_2 \sin x \cos^{-3} x$$

$$\Rightarrow u_p' \cos(x) - 2u_p \sin x = k_2' \cos^{-1} x + 2k_2 \sin x \cos^{-2} x - 2k_2 \cos^{-2} x \sin x = k_2' \cos^{-1} x = \tan x$$

$$\Rightarrow k_2' = \sin x \Rightarrow k_2 = -\cos x$$

$$\Rightarrow u_p = \frac{-\cos x}{\cos^2 x} = -\frac{1}{\cos x} \Rightarrow u = \frac{c}{\cos^2 x} - \frac{1}{\cos x} \Rightarrow k_1'(x) = \frac{c}{\cos^2 x} - \frac{1}{\cos x}$$

$$\Rightarrow k_1(x) = \int \left(\frac{c}{\cos^2 x} - \frac{1}{\cos x}\right) dx = c\tan x - \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x}$$

$$\Rightarrow y_p = k_1(x)\cos(x) = \left(c\tan x - \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x}\right)\cos(x) = c\sin x - \frac{\cos(x)}{2}\ln\frac{1+\sin x}{1-\sin x} \Rightarrow$$

$$\boxed{y = y_h + y_p = c_1 \cos x + c_2 \sin x - \frac{\cos x}{2}\ln\frac{1+\sin x}{1-\sin x}}$$

2. $y'' + 2y' - 2y = e^x + x$

Soit l'équation homgène : $y_h'' + 2y_h' - 2y_h = 0 \Rightarrow r^2 + 2r - 2 = 0 \Rightarrow r \in \left\{-1+\sqrt{3}, -1-\sqrt{3}\right\}$

$\Rightarrow y_h = c_1 e^{\left(-1+\sqrt{3}\right)x} + c_2 e^{\left(-1-\sqrt{3}\right)x}$

$u_1 = ae^x \Rightarrow u_1' = ae^x \Rightarrow u_1'' = ae^x \Rightarrow (a + 2a - 2a)\, e^x = ax = e^x \Rightarrow a = 1$

$u_2 = bx + c \Rightarrow u_2' = b \Rightarrow u_2'' = 0 \Rightarrow (2b - 2bx - 2c) = x \Rightarrow b = -\frac{1}{2}$ *et* $c = -\frac{1}{2}$

$\Rightarrow y_p = u_1 + u_2 = e^x - \frac{1}{2}(x+1) \Rightarrow y = y_h + y_p$

$$\boxed{y = c_1 e^{\left(-1+\sqrt{3}\right)x} + c_2 e^{\left(-1-\sqrt{3}\right)x} + e^x - \frac{1}{2}(x+1)}$$

3. $y'' = y + \frac{1}{\cos x}$,

Soit l'equation homogene : $y_h'' - y_h = 0$

$r^2 - 1 = 0 \Leftrightarrow r \in \{-1, 1\} \Rightarrow y_h = c_1 e^x + c_2 e^{-x}$

Soit $y_1 = k_1(x)\, e^x$ *la solution de l'equation non homogene* $y_p'' = y_p + \frac{1}{\cos x}$

$\Rightarrow y_1' = e^x k_1' + k_1 e^x \Rightarrow y_1'' = e^x k_1' + e^x k_1'' + k_1 e^x + k_1' e^x = e^x k_1'' + 2k_1' e^x + k_1 e^x$

$\Rightarrow y_1'' - y_1 = e^x k_1'' + 2k_1' e^x = \frac{1}{\cos x} \Rightarrow k_1'' + 2k_1' = \frac{e^{-x}}{\cos x}$

Posons $u_1 = k_1' \Rightarrow u_1' + 2u_1 = \frac{e^{-x}}{\cos x}$

Soit l'equation homogene $u_1' + 2u_1 = 0 \Rightarrow \frac{du_1}{dx} = -2u_1 \Rightarrow \int \frac{du_1}{u_1} = -2\int dx = -2x + c_3$

$\Rightarrow \ln|u_1| = c_3 - 2x \Rightarrow |u_1| = e^{c_3 - 2x} = e^{c_3} e^{-2x} \Rightarrow u_1 = \pm e^{c_3} e^{-2x} = c_4 e^{-2x}$

Soit $u_1(x) = b(x)\, e^{-2x} \Rightarrow u_1' = b' e^{-2x} - 2be^{-2x}$

$\Rightarrow u_1' + 2u_1 = b' e^{-2x} - 2be^{-2x} + 2be^{-2x} = b' e^{-2x} = \frac{e^{-x}}{\cos x} \Rightarrow b' = \frac{e^x}{\cos x} \Rightarrow b(x) = \int \frac{e^x}{\cos x} dx$

$\Rightarrow u_1(x) = \left(\int \frac{e^x}{\cos x} dx\right) e^{-2x} \Rightarrow u_1' = \left(\int \frac{e^x}{\cos x} dx\right) e^{-2x} \Rightarrow u_1 = -\frac{1}{2}\left(e^{-2x}\int \frac{e^x}{\cos x} dx - e^{-2x}\frac{e^x}{\cos x}\right) + c$

$\Rightarrow y_1 = \left[-\frac{1}{2}\left(e^{-2x}\int \frac{e^x}{\cos x} dx - e^{-2x}\frac{e^x}{\cos x}\right) + c\right] e^x = -\frac{1}{2}\left(e^{-x}\int \frac{e^x}{\cos x} dx - \frac{1}{\cos x}\right) + c_1 e^x$

Soit $y_2 = k_2(x)\, e^x$ *la solution de l'equation non homogene* $y_p'' = y_p + \frac{1}{\cos x}$

$\Rightarrow y_2 = -\frac{1}{2}\left(-e^x \int \frac{e{-x}}{\cos x} dx + \frac{1}{\cos x}\right) + c_2 e^{-x} \Rightarrow y = y_1 + y_2 = c_1 e^x + c_2 e^{-x} - \frac{1}{2}\left(e^{-x}\int \frac{e^x}{\cos x} dx - e^x \int \frac{e^{-x}}{\cos x} dx\right)$

$$\boxed{y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}\left(e^{-x}\int \frac{e^x}{\cos x} dx - e^x \int \frac{e^{-x}}{\cos x} dx\right)}$$

4. $y'' + 2y' + y = \frac{e^{-x}}{x}$,

Soit l'equation homogène : $y_h'' + 2y_h' + y_h = 0$

$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r_1 = r_2 = -1 \Rightarrow y_h = (ax + b)\, e^{-x}$

Soit $y_p = k(x)\, e^{-x}$, *la soluton de l'equation non homogène* $y_p'' + 2y_p' + y_p = \frac{e^{-x}}{x}$

$y_p = k(x)\, e^{-x} \Rightarrow y_p' = k'(x)\, e^{-x} - k(x)\, e^{-x}$

$\Rightarrow y_p'' = k''(x)\, e^{-x} - k'(x)\, e^{-x} - k'(x)\, e^{-x} + k(x)\, e^{-x} = k''(x)\, e^{-x} - 2k'(x)\, e^{-x} + k(x)\, e^{-x}$

$\Rightarrow y'' + 2y' + y = k''(x)\, e^{-x} - 2k'(x)\, e^{-x} + k(x)\, e^{-x} + 2k'(x)\, e^{-x} - 2k(x)\, e^{-x} + k(x)\, e^{-x} = k''(x)\, e^{-x} = \frac{e^{-x}}{x}$

$\Rightarrow k''(x) = \frac{1}{x} \Rightarrow \int k''(x)\, dx = \int \frac{1}{x} dx = \ln|x| + c_1 \Rightarrow k'(x) = \ln x + c_1$

$\Rightarrow k(x) = \int (\ln x + c_1)\, dx = x(c_1 + \ln x - 1) \Rightarrow y_p = k(x)\, e^{-x} = x(c_1 + \ln x - 1)\, e^{-x}$

$\Rightarrow y = (ax + b)\, e^{-x} + x(c_1 + \ln x - 1)\, e^{-x} = be^{-x} + +x(a + c_1 + \ln x - 1)\, e^{-x} = be^{-x} + x(c + \ln x)\, e^{-x}$

$$\boxed{y = [b + x(c + \ln x)]\, e^{-x}}$$

5. $y'' + 4y = \sin x,$

<u>Première méthode :</u>

Soit l'equation homogène : $y_h'' + 4y_h = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r \in \{2i, -2i\} \Rightarrow y_h = c_1 \cos 2x + c_2 \cos 2x.$

$y_1 = \cos 2x$ *et* $y_2 = \sin 2x$ *et* $f(x) = \sin x$

$y_p = u_1 \cos 2x + u_2 \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ \frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos x \end{vmatrix} = 2\cos^2 x + 2\sin^2 x = 2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & \frac{\partial y_2}{\partial x} \end{vmatrix} = \begin{vmatrix} 0 & \sin 2x \\ \sin x & 2\cos x \end{vmatrix} = -\sin x \sin 2x$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ \frac{\partial y_1}{\partial x} & f(x) \end{vmatrix} = \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \sin x \end{vmatrix} = \sin x \cos 2x$$

$\Rightarrow u_1' = \frac{-\sin x \sin 2x}{2} \Rightarrow u_1 = -\int \sin x^2 \cos x dx = -\int \sin^2 x d(\sin x) = -\frac{1}{3}\sin^3 x$

$\Rightarrow u_2' = \frac{\sin x \cos 2x}{2} \Rightarrow u_2 = \int \sin x \cos^2 x dx - \frac{1}{2}\int \sin x dx$

$= -\int \cos^2 x d(\cos x) - \frac{1}{2}\int \sin x dx = -\frac{1}{3}\cos^3 x + \frac{1}{2}\cos x$

$\Rightarrow y_p = u_1 \cos 2x + u_2 \sin 2x$

$= \left(-\frac{1}{3}\sin^3 x\right)\cos 2x + \left(-\frac{1}{3}\cos^3 x + \frac{1}{2}\cos x\right)\sin 2x$

$= -\frac{1}{3}\sin^3 x\left(2\cos^2 x - 1\right) + 2\left(-\frac{1}{3}\cos^3 x + \frac{1}{2}\cos x\right)\sin x \cos x$

$= -\frac{2}{3}\sin^3 x \cos^2 x + \frac{1}{3}\sin^3 x - \frac{2}{3}\cos^4 x \sin x + \cos^2 x \sin x$

$= -\frac{2}{3}\sin x \cos^2 x\left(\sin^2 x + \cos^2 x\right) + \frac{1}{3}\sin^3 x + \cos^2 x \sin x$

$= -\frac{2}{3}\sin x \cos^2 x + \cos^2 x \sin x + \frac{1}{3}\sin^3 x = \frac{1}{3}\cos^2 x \sin x + \frac{1}{3}\sin^3 x$

$= \frac{1}{3}\sin x\left(\cos^2 x + \sin^2 x\right) = \frac{1}{3}\sin x$

$\Rightarrow y = y_h + y_p = \frac{1}{3}\sin x + c_1 \cos 2x + c_2 \cos 2x,$ *d'où*

$$\boxed{y = \frac{1}{3}\sin x + c_1 \cos 2x + c_2 \cos 2x}$$

<u>Deuxiène méthode :</u>

$y'' + 4y = \sin x \Rightarrow y_p = a\sin x + b\cos x + c \Rightarrow y_p' = a\cos x - b\sin x \Rightarrow$

$\Rightarrow y_p'' = -a\sin x - b\cos x \Rightarrow y_p'' + 4y = 3a\sin x + 3b\cos x + 4c = \sin x$

$$\Rightarrow \begin{cases} 3a = 1 \\ 3b = 0 \\ 4c = 0 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{3} \\ b = 0 \\ c = 0 \end{cases} \Rightarrow y_p = \frac{1}{3}\sin x$$

$\Rightarrow y = y_h + y_p = \frac{1}{3}\sin x + c_1 \cos 2x + c_2 \cos 2x,$ *d'où*

$$\boxed{y = \frac{1}{3}\sin x + c_1 \cos 2x + c_2 \cos 2x}$$