Chapter 03

Real-Valued Functions of a Real Variable

3.1 Generalities

Let $D \subset \mathbb{R}$. A function f of a real variable is a rule which associates with each $x \in D$ one and only one $y \in \mathbb{R}$.

Notation: $f : D \to \mathbb{R}$.

D is called the domain of the function.

Example

1) $f :]0, \infty[\to \mathbb{R}$

$$x \longmapsto \ln(x)$$

2) $g :]1, \infty[\to \mathbb{R}$

$$x \longmapsto \frac{1}{1-x}$$

Graph of function

Graph of function f is a set of ordered pairs of real numbers $(x, f(x))$, where $x \in D(f)$. We write

$graph\ f = \{(x, f(x))\ /\ x \in D(f)\}$.

Monotone functions

Function $f(x)$ defined on the set D is called

increasing, if

$$\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) < f(x_2).$$

non-increasing, if

$$\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2).$$

decreasing, if

$$\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) > f(x_2).$$

non-decreasing, if

$$\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2).$$

The above functions are said to be monotone on D,

incresing and **decresing** functions are said to be **strictly monotone**.

Proposition: A sum of two increasing (decreasing) functions is an increasing (decreasing) function.

Even and odd functions

Let function $f(x)$ be defined on the set D, which contains with any number x also number $-x$.

•Function $f(x)$ is said to be **even** on D, if

$$\forall x \in D \Longrightarrow f(-x) = f(x)$$

•Function $f(x)$ is said to be **odd** on D, if

$$\forall x \in D \Longrightarrow f(-x) = -f(x)$$

Graph of an **even** function is **symmetric with respect to** the $y-$axis, while graph of an **odd** function is **symmetric** with respect to the **origin O** of the coordinate system.

Periodic function

Let function $f(x)$ be defined on the set D and p be a positive real number.

Function f is called **periodic** with the period p, if

1. for any $p \in D$ also number $x \pm p \in D$.
2. for all $x \in D$ holds $f(x \pm p) = f(x)$.

Bounded function

Function $f(x)$ defined on the set $D(f)$ is called bounded (bounded above, bounded below), iff there exists a real number K such that, for all x from $D(f)$ holds:

$$|f(x)| \leq K$$

It means, that a function is **bounded** (bounded above, bounded below), if its **range** $R(f)$ is a **bounded** (bounded above, bounded below) set of real numbers.

Function that is **not bounded** is called **unbounded** function

Operations on functions

Let $f, g : D \to \mathbb{R}$

1. $(f \pm g)(x) = f(x) + g(x)$.
2. $(f.g)(x) = f(x).g(x)$.
3. $(\dfrac{f}{g})(x) = \dfrac{f(x)}{g(x)}, \;\; g(x) \neq 0$.

3.2 Limits of function

Définition 3.2.1

Let $f : D \to \mathbb{R}$ be a function. Let $x_0 \in D$. Then, $L \in \mathbb{R}$ is called the limite of f as x approaches x_0, if for any $\varepsilon > 0$, we can find $\delta > 0$ such that

$$\forall x, |x - x_0| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

We will write

$$\lim_{x \to x_0} f(x) = L$$

3.2.1 One-sided Limits

Right-Hand Limit

The right-hand limit of f at x_0 is L, denoted by

$$\lim_{x \to x_0^+} f(x) = L, or(\lim_{x \overset{>}{\to} x_0} f(x) = L)$$

Left-Hand Limit

The left-hand limit of f at x_0 is L, denoted by

$$\lim_{x \to x_0^-} f(x) = L, or \lim_{x \overset{<}{\to} x_0} f(x) = L$$

If The right-hand and left-hand limits coincide, we say the common value as the limit of $f(x)$ at x_0 and denote it by

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} f(x) = L$$

Proposition

The limit of a function is unique if it exists.

Examples

1) $f : \mathbb{R}/\{0\} \to \mathbb{R}$

$x \longmapsto f(x) = \frac{1}{x}$

The limit, $\lim_{x \to 0} f(x)$ doesn't exists, because $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = +\infty$, and

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty.$

$\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x).$

2) $g : \mathbb{R} \to \mathbb{R}$

$x \longmapsto g(x) = |x|$

-Compute $\lim_{x\to 0} g(x)$.

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x = 0$, and $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-}(-x) = 0$

Since $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = 0$. Then, g has a limit at 0.

Properties

$f, g : D \to \mathbb{R}$, if $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} g(x) = L_2$, then

- $\lim_{x\to x_0} [\alpha f(x) + \beta g(x)] = \alpha \lim_{x\to x_0} f(x) + \beta \lim_{x\to x_0} g(x) = \alpha L_1 + \beta L_2$, $(\forall \alpha, \beta \in \mathbb{R})$.
- $\lim_{x\to x_0} [f(x)g(x)] = (\lim_{x\to x_0} f(x))(\lim_{x\to x_0} g(x)) = L_1 L_2$.
- $\lim_{x\to x_0}(\dfrac{f(x)}{g(x)}) = \dfrac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \dfrac{L_1}{L_2}$, if $L_2 \neq 0$.

Theorem(Squeeze Theorem)

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that

- $g(x) \leq f(x) \leq h(x)$ for all $x \neq x_0$
- $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = L$

Then, $\lim_{x\to x_0} f(x) = L$.

Example

Show $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$

We have

$$-1 \leq \sin(\tfrac{1}{x}) \leq 1 \Longrightarrow -x^2 \leq x^2 \sin(\tfrac{1}{x}) \leq x^2, \text{ for all } x \neq 0.$$

$\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$

Therefore, by **Squeeze Theorem**, we conclude that $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$.

Indeterminate Forms (I.F):

$\dfrac{0}{0}$; $\dfrac{\infty}{\infty}$; $0.\infty$; $\infty - \infty$; 0^∞; ∞^0; 1^∞.

Now let us discuss these forms one by one:

Inditerminate From $\frac{0}{0}$ and $\frac{\infty}{\infty}$(Hospital).

Inditerminate From $0.\infty$; transformation to $\frac{0}{0}$. Then it becomes,

$\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{f(x)}{\frac{1}{g(x)}}$.

or transformation to $\frac{\infty}{\infty}$.Then it becomes, $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{g(x)}{\frac{1}{f(x)}}$.

Inditerminate From 1^∞; $(\lim_{x\to x_0} f(x) = 1, \lim_{x\to x_0} g(x) = \infty)$. Transformation to $\frac{0}{0}$. Then it becomes, $\lim_{x\to x_0}(f(x))^{g(x)} = \exp(\lim_{x\to x_0} \frac{\ln f(x)}{\frac{1}{g(x)}})$.

Inditerminate From $0^0(\lim_{x\to x_0} f(x) = 0^+, \lim_{x\to x_0} g(x) = 0)$. Transformation to $\frac{0}{0}$. Then it becomes, $\lim_{x\to x_0}(f(x))^{g(x)} = \exp(\lim_{x\to x_0} \frac{g(x)}{\frac{1}{\ln f(x)}})$.

Inditerminate From $\infty^0, (\lim_{x\to x_0} f(x) = \infty, \lim_{x\to x_0} g(x) = 0)$. Transformation to $\frac{0}{0}$. Then it becomes, $\lim_{x\to x_0}(f(x))^{g(x)} = \exp(\lim_{x\to x_0} \frac{g(x)}{\frac{1}{\ln f(x)}})$.

Inditerminate From $\infty - \infty, (\lim_{x\to x_0} f(x) = \infty, \lim_{x\to x_0} g(x) = \infty)$. Transformation to $\frac{0}{0}$.

Then it becomes, $\lim_{x\to x_0}(f(x) - g(x)) = \lim_{x\to x_0}(\frac{(\frac{1}{g(x)} - \frac{1}{f(x)})}{\frac{1}{f(x)g(x)}})$.

Theorem (L'Hopital's Rule)

For a $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ of the Indeterminate Form $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f^{'}(x)}{g^{'}(x)}$$

Examples

1) L'Hopital's Rule and Indeterminate Form $\frac{0}{0}$

Compute $\lim_{x\to\pi} \frac{x^2-\pi^2}{\sin x}$.

Solution: we use L'Hopital's Rule. Since the numerator and denomirator both approach zero.

$$\lim_{x\to\pi} \frac{x^2 - \pi^2}{\sin x} = \frac{0}{0} \stackrel{H}{=} \lim_{x\to\pi} \frac{2x}{\cos x} = -2\pi.$$

2) L'Hopital's Rule and Indeterminate Form $\frac{\infty}{\infty}$

Compute $\lim_{x\to 0^+} \frac{\frac{1}{x^2}}{\ln x}$.

Solution: $\lim_{x\to 0^+} \frac{\frac{1}{x^2}}{\ln x} = \frac{\infty}{\infty} \stackrel{H}{=} \lim_{x\to 0^+} \frac{\frac{-2}{x^3}}{\frac{1}{x}} \stackrel{H}{=} \lim_{x\to 0^+} \frac{\frac{-2}{x^3}}{\frac{1}{x}}$

$$= \lim_{x\to 0^+} \frac{\frac{-6}{x^4}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{-6}{x^2} = -\infty.$$

3) Indeterminate From $0.\infty$, (transformation to $\frac{\infty}{\infty}$)

Compute $\lim_{x\to 0^+} x\ln x$.

Solution: $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty}$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+}(-x) = 0.$$

4) Indeterminate From 1^∞, (transformation to $\frac{0}{0}$)

Compute $\lim_{x\to 1^+} x^{\frac{1}{x-1}}$

Solution: $L = \lim_{x\to 1^+}(x^{\frac{1}{x-1}}) \Longrightarrow \ln L = \lim_{x\to 1^+} \ln(x^{\frac{1}{x-1}}) = \lim_{x\to 1^+} \frac{\ln x}{x-1} = \frac{0}{0}$.

Therefore, we can apply L'Hopital's Rule:

$$\ln L = \lim_{x\to 1^+} \frac{\ln x}{x-1} \overset{H}{=} \lim_{x\to 1^+} \frac{\frac{1}{x}}{1} = 1.$$

Thus, $\ln L = 1 \Longrightarrow L = \lim_{x\to 1^+} x^{\frac{1}{x-1}} = e$.

5) Indeterminate From $\infty - \infty$, (transformation to $\frac{0}{0}$)

Compute $\lim_{x\to 0^+}(\frac{1}{\sin x} - \frac{1}{x})$

Solution: $\lim_{x\to 0^+}(\frac{1}{\sin x} - \frac{1}{x}) = \frac{x-\sin x}{x\sin x} = \frac{0}{0}$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{1-\cos x}{\sin x + x\cos x} = \frac{0}{0}$$

$$\overset{H}{=} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.$$

Remark: some of the forms which are not indeterminate are:

$\frac{0}{\infty}$; $\frac{1}{\infty}$have limite as 0.

$\infty.\infty$ has limite as ∞.

0^∞ and ∞^∞ have the limites as 0 and ∞ respectively.

3.3 Continuity and IVT

3.3.1 Continuity of Function

Définition 3.3.1 *continuous at a point*

Let $f : D \to \mathbb{R}$ be a function, f is continuous at a point a if

$$\lim_{x \to a} f(x) = f(a).$$

On the other hand, if f is defined on an open interval containing a, except perhaps at a, we can say that f is discontinuous at a if f is not continuous at a.

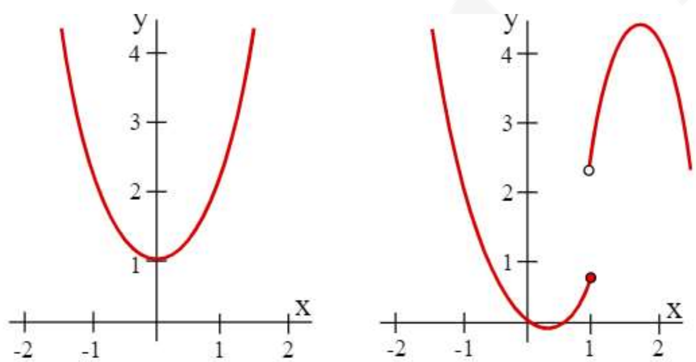


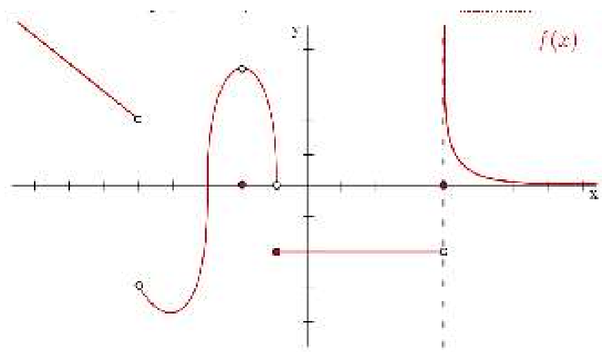
Figure 3.3. (a) A continuous function, (b) A function with a discontinuity at $x = 1$.

Graphically, you can think of continuity as being able to draw your function without having to lift your pencil off the paper. If your pencil has to jump off the page to continue drawing the function, then the function is not continuous at that point. This is illustrated in (figure 3.3-b) where if we tried to draw the function (from left to right) we need to lift our pencil off the page once we reach the point $x = 1$ in order to be able to continue drawing the function.

Définition 3.3.2 *continuous on an open interval*

A function f is continuous on an open interval $]a, b[$ if it is continuous at every point in the interval.

Furthermore, a function is everywhere continuous if it is continuous on the entire real number line $]-\infty, +\infty[$.



The function f is discontinuities at $x = -5, x = -2, x = -1$, and $x = 4$.

Définition 3.3.3 *continuous from the Right and from the Left*

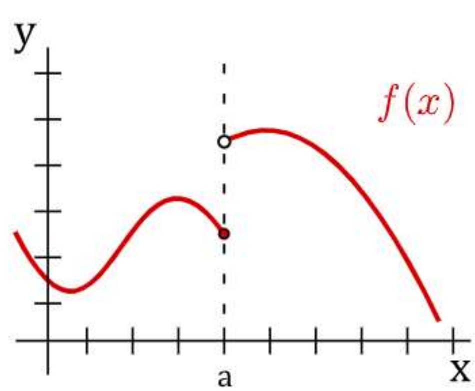
A function f is right continuous at a point a if

$$\lim_{x \to a^+} f(x) = f(a)$$

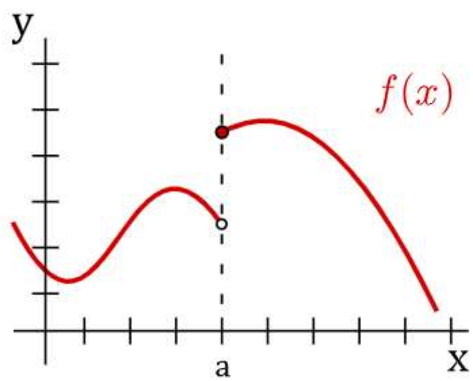
and left continuous at a point a if

$$\lim_{x \to a^-} f(x) = f(a)$$

If a function f is continuous at a then it is both left and right continuous at a.



$\lim_{x \to a^-} f(x) = f(a)$(left continuous at a point a)



$\lim_{x\to a^+} f(x) = f(a)$(right continuous at a point a)

Example 01

$$f(x) = \begin{cases} x-2, & x \leq 0 \\ 2\ , & x > 0 \end{cases}$$

Away from $x = 0$, we see that f is continuous. Therefore, we look at $x = 0$.

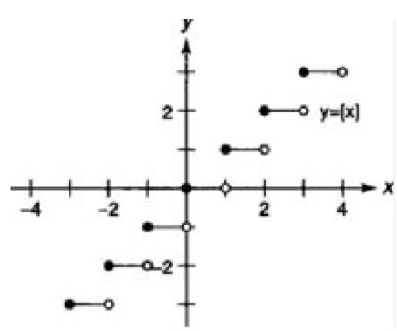
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2 = 2$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x - 2 = -2$

$\lim_{x\to 0}$ does not exist. We conclude that f is discontinuous at $x = 0$.Therefore the function f is continuous on $\mathbb{R} - \{0\}$.

Example 02

The floor function $f(x) = [x]$ is continuous in every open interval between integers, $]n, n+1[$ for any integer n. However, it is not continuous at any integer n.



Remark

The following functions are all continuous at all points of there domains:

(i) Polynomials;(ii) Rational Functions;(iii) Root Functions;(iv) Trigonometric Functions

(v) Inverse Trigonometric Functions; (vi) Exponential Functions; (vii) Logarithmic Functions

Exercise 01

Determine the values of x for which each function is continuous.

1) $f(x) = \frac{2}{x^2+1}$.

Soltion: Since $x^2+1=0$ has no real solutions, we see that f is continuous for all $x \in \mathbb{R}$.

2) $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$.

Soltion

$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x\to 1} x+1 = 2$.Since $\lim_{x\to 1} f(x) = 2 = f(1)$, f is continuous at $x=1$. Additionally, we see that f is continuous everywhere else. (because it is elementary function).

Hence, f is continuous on $\mathbb{R}$.

Exercise 02

Consider the function

$g(x) = \begin{cases} 2x-3, & \text{if } x < 1 \\ 0, & \text{if } x \geq 1 \end{cases}$

Show that it is continuous at the point $x=1$. Is g a continuous function?.

Solution

$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+}(0) = 0$,

and

$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-}(2x-3) = -1$.

Since $\lim_{x\to 1^+} g(x) \neq \lim_{x\to 1^-} g(x)$. Hence g is not continuous at $x=1$ and is not a continuous function.

Exercise 03

Find the values of α that make the function $f(x)$ continuous for all real numbers.

$$f(x) = \begin{cases} 4x+5, & \text{if } x \geq -2 \\ x^2+\alpha, & \text{if } x < -2 \end{cases}$$

Solution

First, we note that, for $x > -2$, $f(x) = 4x+5$ is continuous. For $x < -2$, $f(x) = x^2+\alpha$ will be continuous for all choices of α. We now look at $x = 2$.

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+}(4x+5) = -3$.

$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-}(x^2+\alpha) = 4+\alpha$

In order for the limit to exist, we therefore require $4+\alpha = -3 \Longrightarrow \alpha$ has to be -7.

With this choice of we get that

$$\lim_{x\to -2} f(x) = f(-2) = -3$$

and so would be continuous. For all other choices of α, f would be discontinuous at $x = -2$.

Operations of Continuous Functions

If f and g are continous at a and λ is a constant, then the following functions are also continuous at a.

$f \pm g;\ \ \lambda f;\ \ fg;\ \ \frac{f}{g}(g \neq 0)$

Continuity of Function Composition

If g is continuous at a and f is continuous at $g(a)$, then the composition function $f \circ g$ is continuous at a.

Example

$g(x) = x^2$ is continuous on $\mathbb{R}$ since it is a polynomial, and $f(x) = \cos(x)$ is also continuous everywhere. Therefore $(f \circ g)(x)$ is continuous on $\mathbb{R}$.

Proposition

If $f(x)$ is continuous at b and $\lim_{x\to a} g(x) = b$ then,

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

Example Evaluate the following limit.

$$\lim_{x\to 0} e^{\sin x}$$

Since we know that exponentials are continuous everywhere we can use the proposition above.

$$\lim_{x\to 0} e^{\sin(x)} = e^{\lim_{x\to 0} \sin x} = e^0 = 1.$$

3.3.2 Theorem Intermediate Value Theorem (IVT)

If f is continuous on the interval $[a, b]$ and N is between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$,then there is a number c in $]a, b[$ such that $f(c) = N$.

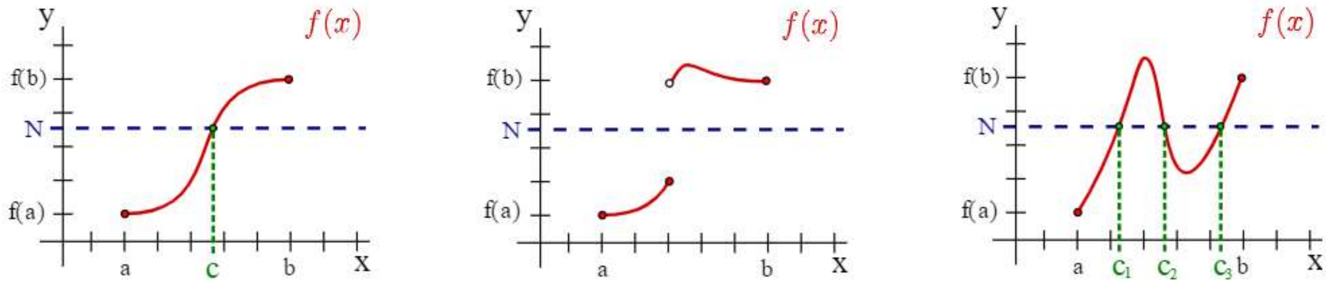
f is continuous on $[a, b]$ and $f(a) \leq N \leq f(b) \Longrightarrow \exists c \in]a, b[: f(c) = N$

The Intermediate Value Theorem guarantees that if $f(x)$ is continuous and

$f(a) \leq N \leq f(b)$.The line $y = N$ intersects the function at some point $x = c$. Such a number is between a and b and has the property that $f(c) = N$.(See Figure 3.5 (a)). We can also think of the theorem as saying if we draw the line $y = N$ between the lines $y = f(a)$ and $y = f(b)$, then the function cannot jump over the line $y = N$. On the other hand, if $f(x)$ is not continuous, then the theorem may not hold. (See Figure 3.5 (b)) where there is no number c in $]a, b[$ such that $f(c) = N$.

Finally, we remark that there may be multiple choices for c (i.e., lots of numbers betwee a and b with $y-$coordinat N). (See Figure 3.5 (c)) for such an example.

Figure 3.5. (a) A continuous function where IVT holds for a single value c. (b) A discontinuous function where IVT fails to hold. (c) A continuous function where IVT holds for multiple values in $]a, b[$.



Application of Intermediate Value Theorem

The important application of the intermediate value theorem is to verify the existence of a root of an equation in a given interval. In particular, the IVT theorem is used to see whether a given function has its zero ($f(x)=0$) within the given interval $]a,b[$. To verify this, we follow the steps below:

Step 1: Find $f(a)$ and $f(b)$.

Step 2: If $f(a)<0<f(b)$) (i.e., $f(a)$ is negative and $f(b)$ is positive).

Then $f(x)$ has a zero (i.e., $f(x)=0$) in the interval $]a,b[$.

Let us examine the following example.

Example 01: Use the IVT to show that the function $f(x)=x^3+2x-1$ has a zero in the interval $[0,1]$.

Solution

$f(x)$ is continuous everywhere as it is a polynomial function; thus, $f(x)$ is clearly continuous on $[0,1]$.

Further, $f(0)=-1$ and $f(1)=2$ and $-1<0<2$.

By the Intermediate Value Theorem, there must exist a $c\in[0,1]$ such that $f(c)=0$.

Example 02 Explain why the function $f=x^3+3x^2+x-2$ has a root between 0 and 1.

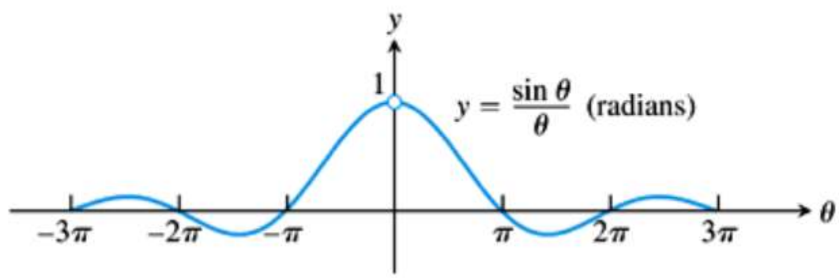
Solution

by the IVT, f is continuous. Since $f(0)=-2$ and $f(1)=3$, and $-2<0<3$, there is a $c\in\,]0,1[$ such that $f(c)=0$.

3.3.3 Continuous extension to a point

Example

$$f(x) = \frac{\sin x}{x}$$



is defined and continuous for all $x \neq 0$. As $\lim_{x\to 0} \frac{\sin x}{x} = 1$, it makes sense to define a new function

$$F(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{for } x \neq 0 \\ 1, & \text{for } x = 0 \end{cases}$$

Définition 3.3.4 *Let $f : D - \{c\} \to \mathbb{R}$ be a function*

If $\lim_{x\to c} f(x) = L$ exists, but $f(c)$ is not defined, we define a new function

$$F(x) = \begin{cases} f(x), & \text{for } x \neq c \\ L, & \text{for } x = c \end{cases}$$

which is continuous at c. It is called the **continuous extension** of $f(x)$ to c.