Distributions analyse de Fourier

Correction de l'examen final

" Février 2022 "

Exercice 1. Voir le cours.

Exercice 2. (1) • H et T sont linéaire (simple)

$$\int_a^b |H(x)|\,dx = \begin{cases} 0 , & a<b\leq 0 \\ b , & a<0\leq b \\ b-a , & 0\leq a<b \end{cases}$$

$$\int_a^b |T(x)|\,dx = \begin{cases} \frac{1}{2}(a^2-b^2) , & a<b\leq 0 \\ \frac{1}{2}(a^2+b^2) , & a<0\leq b \\ \frac{1}{2}(b^2-a^2) , & 0\leq a<b \end{cases}$$

H et $T \in L^1_{Loc}(\mathbb{R})$ donc $H, T \in \mathcal{D}'(\mathbb{R})$.

• $\langle H', \varphi\rangle = -\langle H, \varphi'\rangle = -\int_0^\infty \varphi'(t)\,dt = \varphi(0) = \langle \delta, \varphi\rangle$

$\forall \varphi \in \mathcal{D}(\mathbb{R})$ donc $\underline{H' = \delta}$.

• Comme x et $x^2 \in C^\infty(\mathbb{R})$ alors on a

$$(xH)' = (x)'H + xH' = H + x\delta$$

mais $\langle x\delta, \varphi\rangle = \langle \delta, x\varphi\rangle = 0\,\varphi(0) = 0 = \langle 0, \varphi\rangle$

donc $\underline{x\delta = 0}$, d'où $\underline{(xH)' = H}$.

$$(x^2H)'' = (2xH + x^2H')' = 2H + 4xH' + x^2H''$$

$$xH' = x\delta = 0 \ ; \ x^2H'' = x^2\delta' = (x^2\delta)' - 2x\delta$$
$$= 0' - 0 = 0 , \quad \underline{(x^2H)'' = 2H}.$$

- $T' = \begin{cases} +1, & si\ x > 0 \\ -1 & si\ x \leq 0 \end{cases} = sgn\, x$

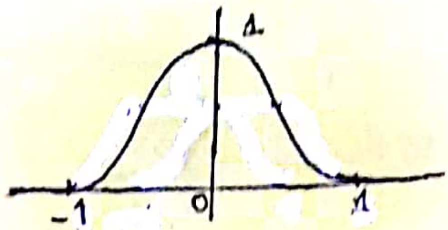
voir le cours.

(2) • $(TH)'$ n'existe pas, car TH n'est pas défini.

- supp $\delta' = \{0\}$ donc $\delta' \in \mathcal{E}'(\mathbb{R})$ et on a

$$\delta' * T = (\delta * T)' = \delta * T' = T' = sgn\, x.$$

Exercice 3.



(1) $\varphi \in \mathcal{D}(\mathbb{R})$ voir le cours

(2) • Si $\frac{|x|}{2} < \frac{1}{2} \Rightarrow |x| < 1$

on a $g\left(\frac{|x|}{2}\right) = 1$ donc $\psi(x) = e^{1 - \frac{1}{1-x^2}} = \varphi(x)$

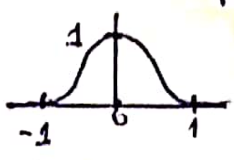
- si $\frac{|x|}{2} \geq \frac{1}{2} \Rightarrow |x| \geq 1$ on a $g\left(\frac{|x|}{2}\right) = 0$ donc

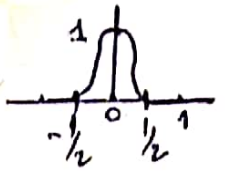
$\psi(x) = 0 = \varphi(x)$. Conclusion $\psi = \varphi$ et $u \equiv 0$.

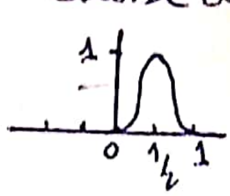
(3) $-1 < 2x - 1 < 1 \Rightarrow 0 < x < 1$, donc il suffit de tracer φ sur $[0,1]$. On peut aussi écrire

$$f(x) = \varphi(2(x - 1/2)) = \tau_{1/2}\, \varphi(2x)$$

courbe de φ — courbe de $\varphi(2\cdot)$ — translation — courbe de f :

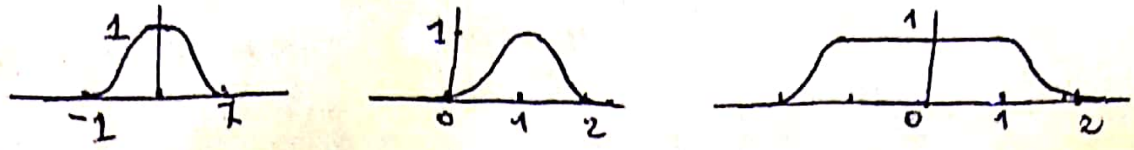




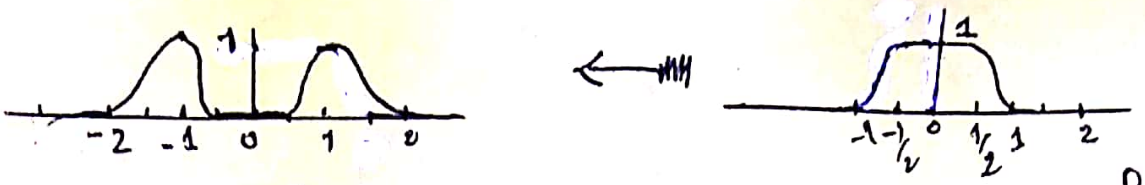


(4) • $x \geq 0$: $h(x+1) = \varphi(x)$

on pose $X = x+1$, alors $h(X) = \varphi(X-1)$: $X \geq 1$



courbe de φ — courbe de $\varphi(x-1)$ — courbe de h



courbe de β — courbe de $h(2\,\cdot)$

(5) • $f(x) = j\, e^{-(j+1)^2 x^2} \in L^1_{Loc}(\mathbb{R})$ donc $T_j \in \mathcal{D}'(\mathbb{R})$

(f est une fonction continue donc dans $L^1_{Loc}(\mathbb{R})$).

• $\langle T_j, \varphi \rangle = j \int_{-\infty}^{\infty} e^{-(j+1)^2 x^2} \varphi(x)\, dx = \frac{j}{j+1} \int_{-\infty}^{\infty} e^{-y^2} \varphi\left(\frac{y}{j+1}\right) dy$

$$\xrightarrow[j \to \infty]{} \int_{-\infty}^{\infty} e^{-y^2} dy\, \varphi(0) = \sqrt{\pi} \langle \delta, \varphi \rangle = \langle \sqrt{\pi}\delta, \varphi \rangle$$

d'où $\underline{\lim_{j \to \infty} T_j = \sqrt{\pi}\,\delta}$.

• $\langle T_j', \varphi \rangle = -\langle T_j, \varphi' \rangle \longrightarrow -\langle \sqrt{\pi}\delta, \varphi' \rangle$
$= \langle \sqrt{\pi}\delta', \varphi \rangle$

donc $\underline{\lim_{j \to \infty} T_j' = \sqrt{\pi}\,\delta'}$.

(6) • Comme $\int_a^b |x|^{-1/2}\,dx < +\infty$

$$= \begin{cases} 2(\sqrt{b}-\sqrt{a}) & \text{si } 0 \le a < b \\ 2(\sqrt{b}+\sqrt{-a}) & \text{si } a \le 0 < b \\ 2\sqrt{-a}-\sqrt{-b} & \text{si } a < b \le 0 \end{cases} \quad \text{donc } T_f \in \mathcal{D}'(\mathbb{R})$$

• $\langle T_f', \varphi\rangle = -\langle T_f, \varphi'\rangle = -\lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty x^{-1/2}\varphi'(x) - \lim_{\varepsilon\downarrow 0} \int_{-\infty}^{-\varepsilon} (-x)^{-1/2}\varphi'(x)$

On a $\int_\varepsilon^\infty x^{-1/2}\varphi'(x)\,dx + \int_{-\infty}^{-\varepsilon} (-x)^{-1/2}\varphi'(x)\,dx$

$$= \frac{\varphi(-\varepsilon)-\varphi(\varepsilon)}{\varepsilon^{1/2}} + \frac{1}{2}\int_\varepsilon^\infty x^{-3/2}\varphi(x)\,dx - \frac{1}{2}\int_{-\infty}^{-\varepsilon} (-x)^{-3/2}\varphi(x)\,dx$$

Or $\frac{\varphi(-\varepsilon)-\varphi(\varepsilon)}{\varepsilon^{1/2}} = -\varepsilon^{1/2}\left(\varphi'(\theta\varepsilon)+\varphi'(-\theta\varepsilon)\right) \longrightarrow 0$ quand $\varepsilon \downarrow 0$

$$\langle T_f', \varphi\rangle = -\frac{1}{2}\int_0^\infty x^{-3/2}\varphi(x)\,dx + \frac{1}{2}\int_{-\infty}^0 (-x)^{-3/2}\varphi(x)\,dx$$

$$T_f' = -\frac{1}{2}\begin{cases} x^{-3/2} & \text{si } x > 0 \\ -(-x)^{-3/2} & \text{si } x \le 0 \end{cases} = -\frac{1}{2}|x|^{-3/2}\begin{cases} 1 & \text{si } x > 0 \\ -1 & \text{si } x \le 0 \end{cases}$$

$$= -\frac{1}{2}|x|^{-3/2}\,\mathrm{sgn}\,x.$$

• $T_k \notin \mathcal{D}'(\mathbb{R})$, car

$$\langle T_k, h\rangle = \int_{-\infty}^\infty \frac{h(x)}{|x|}\,dx \ge \int_0^{1/2} \frac{1}{x}\,dx = \operatorname{Log} x\Big|_0^{1/2} = +\infty$$

n'oublions pas que $h \ge 0$ et $h(x) = 1$ si $0 \le x \le 1$, voir (4).

• $f(x)^2 = |x|^{-1} = k(x)$ donc $T_{f^2} \notin \mathcal{D}'(\mathbb{R})$

Comme $T_{f^2} = T_f \times T_f$, conclusion : le produit dans $\mathcal{D}'(\mathbb{R})$ n'est pas défini.