Divide-and-Conquer Algorithms

Part One

Announcements

- Problem Set One completely due right now. Solutions distributed at the end of lecture.
- Programming section today in Gates B08 from from 3:45PM – 5:00PM.
  - Resumes at normal Thursday schedule (4:15PM – 5:05PM) next week.

Where We've Been

- We have just finished discussing fundamental algorithms on graphs.
- These algorithms are indispensable and show up *everywhere*.
- You can now solve a large class of problems by recognizing that they *reduce* to a problem you already know how to solve.

Where We're Going

- We are about to explore the **divide-and-conquer** paradigm, which gives a useful framework for thinking about problems.
- We will explore several major techniques:
  - Solving problems recursively.
  - Intuitively understanding how the structure of recursive algorithms influences runtime.
  - Recognizing when a problem can be solved by reducing it to a simpler case.

Outline for Today

- **Recurrence Relations**
  - Representing an algorithm's runtime in terms of a simple recurrence.
- **Solving Recurrences**
  - Determining the runtime of a recursive function from a recurrence relation.
- **Sampler of Divide-and-Conquer**
  - A few illustrative problems.

Insertion Sort

- As we saw in Lecture 00, insertion sort can be used to sort an array in time $\Omega(n)$ and $O(n^2)$.
  - It's $\Theta(n^2)$ in the average case.
- Can we do better?

A Better Sorting Algorithm: **Mergesort**

Thinking About O(n^2)

14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4

$T(n)$

2	3	6	7	9	14	15	16

$T(\frac{1}{2}n) \approx \frac{1}{4}T(n)$

1	4	5	8	10	11	12	13

$T(\frac{1}{2}n) \approx \frac{1}{4}T(n)$

```
procedure merge(list A, list B):
   let result be an empty list.
   while both A and B are nonempty:
      if head(A) < head(B):
         append head(A) to result
         remove head(A) from A
      else:
         append head(B) to result
         remove head(B) from B

   append all elements remaining in A to result
   append all elements remaining in B to result

   return result
```

Complexity: $\boldsymbol{\Theta(m + n)}$,
where m and n are the lengths of the input lists.

Motivating Mergesort

- Splitting the input array in half, sorting each half, and merging them back together will take roughly half as long as soring the original array.
- So why not split the array into fourths? Or eighths?
- **Question**: What happens if we *never stop splitting?*

High-Level Idea

- A recursive sorting algorithm!
- **Base Case**:
  - An empty or single-element list is already sorted.
- **Recursive step**:
  - Break the list in half and recursively sort each part.
  - Merge the sorted halves back together.
- This algorithm is called *mergesort*.

```
procedure mergesort(list A):
   if length(A) ≤ 1:
      return A

   let left  be the first  half of the elements of A
   let right be the second half of the elements of A

   return merge(mergesort(left), mergesort(right))
```

$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(n) = T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n)$$

Recurrence Relations

- A **recurrence relation** is a function or sequence whose values are defined in terms of earlier values.
- In our case, we get this recurrence for the runtime of mergesort:

$$\boxed{\begin{aligned} &T(0) = \Theta(1) \\ &T(1) = \Theta(1) \\ &T(n) = T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n) \end{aligned}}$$

- We can **solve** a recurrence by finding an explicit expression for its terms, or by finding an asymptotic bound on its growth rate.
- How do we solve this recurrence?

Simplifying our Recurrence

- It is often difficult to solve recurrences involving floors and ceilings, as ours does.

$$\boxed{\begin{aligned}&T(1) = \Theta(1)\\&T(n) = T(n / 2) + T(n / 2) + \Theta(n)\end{aligned}}$$

- Note that if we only consider $n = 1, 2, 4, 8, 16, \ldots$, then the floors and ceilings are always equivalent to standard division.
- **Simplifying Assumption 1:** We will only consider the recurrence as applied to powers of two.
- We need to justify why this is safe, which we'll do later.

Simplifying our Recurrence

- Without knowing the actual functions hidden by the Θ notation, we cannot get an exact value for the terms in this recurrence.

$$T(1) = c_1$$
$$T(n) = 2T(n / 2) + c_2 n$$

- If the Θ(1) just hides a constant and Θ(*n*) just hides a multiple of *n*, this would be a lot easier to manipulate!

- **Simplifying Assumption 2:** We will pretend that Θ(1) hides some constant and Θ(*n*) hides a multiple of *n*.
- We need to justify why this is safe, which we'll do later.

Simplifying our Recurrence

- Working with two constants c_1 and c_2 is most accurate, but it makes the math a *lot* harder.

$$
\begin{aligned}
&T(1) \leq c \\
&T(n) \leq 2T(n / 2) + cn
\end{aligned}
$$

- If all we care about is getting an asymptotic bound, these constants are unlikely to make a noticeable difference.
- **Simplifying Assumption 3:** Set $c = \max\{c_1, c_2\}$ and replace the equality with an upper bound.
- This is less exact, but is easier to manipulate.

The Final Recurrence

- Here is the final version of the recurrence we'll be working with:

$$\boxed{\begin{aligned}&\mathbf{T(1) \leq \mathit{c}}\\&\mathbf{T(\mathit{n}) \leq 2T(\mathit{n} / 2) + \mathit{cn}}\end{aligned}}$$

- As before, we will justify why all of these simplifications are safe later on.
- The analysis we're about to do (without justifying the simplifications) is at the level we will expect for most of our discussion of divide-and-conquer algorithms.

Getting an Intuition

- Simple recurrence relations often give rise to surprising results.
- It is often useful to build up an intuition for what the recursion solves to before trying to formally prove it.
- We will explore two methods for doing this:
  - The *iteration method*.
  - The *recursion-tree method*.

$$T(1) \le c$$
$$T(n) \le 2T(n/2) + cn$$

$$n / 2^k = 1$$
$$n = 2^k$$
$$\log_2 n = k$$

$$
\begin{aligned}
T(n) &\le 2T\left(\frac{n}{2}\right) + cn \\
&\le 2\left(2T\left(\frac{n}{4}\right) + \frac{cn}{2}\right) + cn \\
&= 4T\left(\frac{n}{4}\right) + cn + cn \\
&= 4T\left(\frac{n}{4}\right) + 2cn \\
&\le 4\left(2T\left(\frac{n}{8}\right) + \frac{cn}{4}\right) + 2cn \\
&= 8T\left(\frac{n}{8}\right) + cn + 2cn \\
&= 8T\left(\frac{n}{8}\right) + 3cn \\
&\dots \\
&\le 2^k T\left(\frac{n}{2^k}\right) + kcn
\end{aligned}
$$

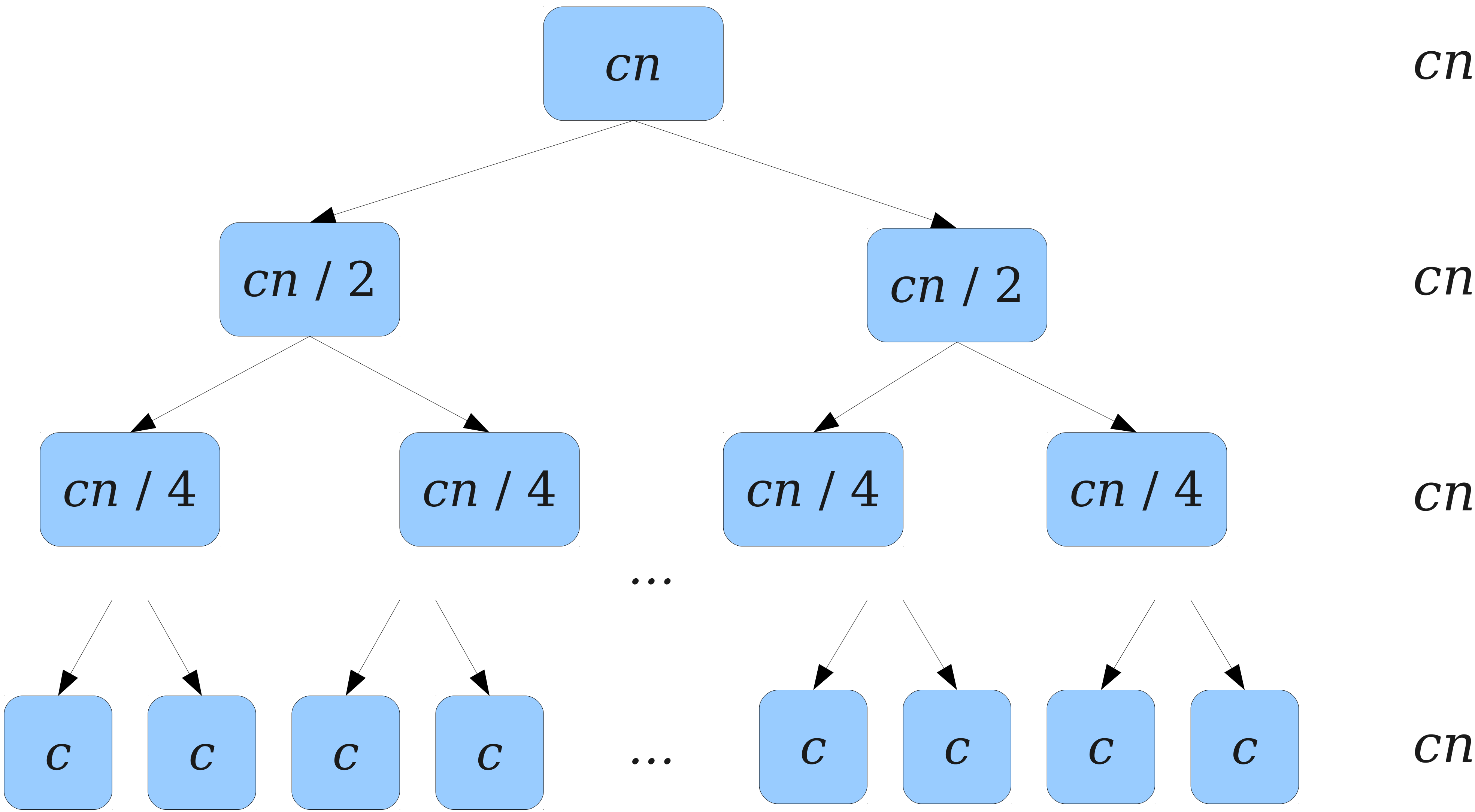
$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le 2T(n/2) + cn \end{array}}$$

$$\begin{aligned} T(n) &\le 2^k T\left(\frac{n}{2^k}\right) + kcn \\ &= 2^{\log_2 n} T(1) + cn\log_2 n \\ &= nT(1) + cn\log_2 n \\ &\le cn + cn\log_2 n \\ &= O(n\log n) \end{aligned}$$

The Iteration Method

- What we just saw is an example of the *iteration method*.
- Keep plugging the recurrence into itself until you spot a pattern, then try to simplify.
- Doesn't always give an exact answer, but useful for building up an intuition.

$$T(1) \le c$$
$$T(n) \le 2T(n / 2) + cn$$



$$cn \log_2 n + cn$$

The Recursion Tree Method

- This diagram is called a **recursion tree** and accounts for how much total work each recursive call makes.
- Often useful to sum up the work across the layers of the tree.

A Formal Proof

- Both the iteration and recursion tree methods suggest that the runtime is at most

$$cn \log_2 n + cn$$

- Neither of these lines of reasoning are perfectly rigorous; how could we formalize this?
- **Induction!**

Theorem: If n is a power of 2, $T(n) \leq cn \log_2 n + cn$

Proof: By induction. As a base case, if $n = 2^0 = 1$, then

$$
\begin{aligned}
T(n) &= T(1) \\
&\leq c \\
&= cn \log_2 n + cn.
\end{aligned}
$$

For the inductive step, assume the claim holds for all $n' < n$ that are powers of two. Then

$$
\begin{aligned}
T(n) &\leq 2T(n / 2) + cn \\
&= 2((cn / 2) \log_2 (n / 2) + cn / 2) + cn \\
&= cn \log_2 (n / 2) + cn + cn \\
&= cn (\log_2 n - 1) + cn + cn \\
&= cn \log_2 n - cn + cn + cn \\
&= cn \log_2 n + cn \quad \blacksquare
\end{aligned}
$$

What This Means

- We have shown that as long as we *only* look at powers of two, the runtime for mergesort is bounded from above by $cn \log_2 n + cn$.

In most cases, it's perfectly safe to stop here and claim we have a working bound. Mergesort is indeed O(*n* log *n*).

- For completeness, let's take some time to see why it is safe to stop here.
- In the future, we won't go into this level of detail.

Replacing Θ

- Our original recurrence was

$$\boxed{\begin{aligned} &T(0) = \Theta(1) \\ &T(1) = \Theta(1) \\ &T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n) \end{aligned}}$$

- We claimed it was safe to remove the Θ notation and rewrite it as

$$\boxed{\begin{aligned} &T(0) \leq c \\ &T(1) \leq c \\ &T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + cn \end{aligned}}$$

- Why can we do this?

Fat Base Cases

- When $n \geq n_0$, we can replace $\Theta(n)$ by cn for some constant c.
- Our simplification in the previous step assumed that $n_0 = 0$. What if this isn't the case?
- Can always rewrite the recurrence to use a “fat base case:”

$$\boxed{\begin{array}{ll} T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + cn & (\text{if } n \geq n_0) \\ T(n) \leq c & (\text{otherwise}) \end{array}}$$

- Makes the induction a *lot* harder to do, but the result would come out the same.

Non Powers of Two

- Consider this recurrence:

$$\boxed{\begin{aligned}&T(0) \le c\\&T(1) \le c\\&T(n) \le T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + cn\end{aligned}}$$

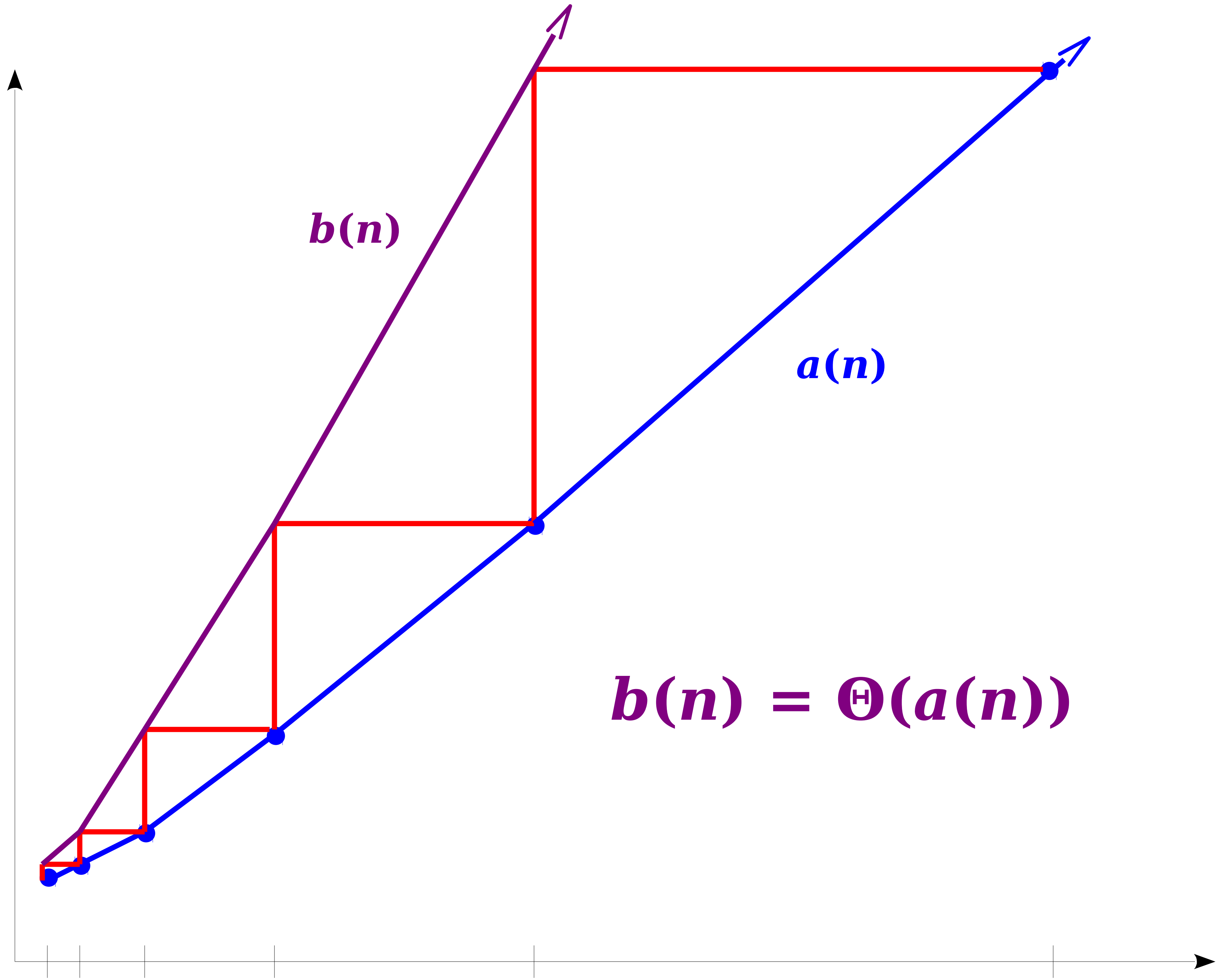
- We know that for powers of two, this is upper bounded by $cn \log_2 n + cn$.
- Does that upper bound still hold for values other than powers of two?
- If not, is our bound even useful?

Non Powers of Two

- Can we claim that since $T(n) \leq cn \log_2 n + cn$ when n is a power of two, that $T(n) = O(n \log n)$?
- Without more work, **no**. Consider this function:

$$f(n) = \begin{cases} n \log_2 n & \text{if } n = 2^k \\ n! & \text{otherwise} \end{cases}$$

- Only looking at inputs that are powers of two, we might claim that $f(n) = \Theta(n \log n)$, even though this isn't the case!
- We need to do extra work to show that $T(n)$ is "well-behaved" enough to extrapolate.



b(n)
a(n)
b(n) = Θ(a(n))

Our Proof Strategy

- We will proceed as follows:
  - Show that the values generated by the recurrence are nondecreasing.
  - For each non power-of-two n, provide an upper bound T(n) using our upper bound on the next power of two greater than n.
  - Show that the upper bound we find this way is asymptotically equivalent (in terms of Θ) to our original bound.

Making Things Easier

- We are given this recurrence:

$$T(0) \leq c$$
$$T(1) \leq c$$
$$T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + cn$$

- This only gives an upper bound on T(n); we don't know the exact values.
- Let's define a new function $f(n)$ as follows:

$$f(0) = c$$
$$f(1) = c$$
$$f(n) = f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn$$

- Note that $T(n) \leq f(n)$ for all $n \in \mathbb{N}$.

$$
\boxed{\begin{aligned}
&f(0) = c \\
&f(1) = c \\
&f(n) = f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn
\end{aligned}}
$$

Lemma: $f(n + 1) \geq f(n)$ for all $n \in \mathbb{N}$.

Proof: By induction on n. As a base case, note that

$$f(1) = c \geq c = f(0)$$

For the inductive step, assume that for some n that the lemma holds for all $n' < n$. Then

$$
\begin{aligned}
f(n + 1) &= f(\lceil (n+1) / 2 \rceil) + f(\lfloor (n+1) / 2 \rfloor) + c(n+1) \\
&\geq f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn \\
&= f(n) \blacksquare
\end{aligned}
$$

***Theorem:* T(n) = O(n log n)**

Proof: Consider any $n \in \mathbb{N}$ with $n \geq 1$. Let k be such that $2^k \leq n < 2^{k+1}$. Thus $2^{k+1} \leq 2n < 2^{k+2}$.

From our lemma, we know that

$$T(n) \leq f(n) \leq f(2^{k+1})$$

Using our upper bound for powers of two:

$$f(2^{k+1}) \leq c(2^{k+1}) \log_2 (2^{k+1}) + c(2^{k+1})$$

Therefore

$$\begin{aligned} T(n) &\leq c(2^{k+1}) \log_2 (2^{k+1}) + c(2^{k+1}) \\ &\leq c(2n) \log_2 (2n) + 2cn \\ &= 2cn (\log_2 n + 1) + 2cn \\ &= 2cn \log_2 n + 4cn \end{aligned}$$

So for any $n \geq 1$, $T(n) \leq 2cn \log_2 n + 4cn$. Thus $T(n) = O(n \log n)$. ■

Summary

- We can safely extrapolate from the runtime bounds at powers of two for the following reasons:
  - The runtime is nondecreasing, so we can use powers of two to provide upper bounds on other points.
  - The runtime grows only polynomially, so this upper bounding strategy does not produce values that are “too much” bigger than the actual values.
- **In the future, we will assume that this line of proof works and will not repeat it.**

Perfectly Safe Assumptions

- For the purposes of this class, you can safely simplify recurrences by
  - Only evaluating the recurrences at powers of some number to avoid ceilings and floors.
  - Replace $\Theta(f(n))$ or $O(f(n))$ terms in a recurrence with a constant multiple of $f(n)$.
  - Replace all constants with a single constant equal to the max of all of the constants.

A Different Problem:

Maximum Single-Sell Profit

Maximum Single-Sell Profit

13	17	15	8	14	15	19	7	8	9

```
procedure maxProfit(list prices):
  if length(prices) ≤ 1:
     return 0

  let left  be the first  half of prices
  let right be the second half of prices

  return max(maxProfit(left), maxProfit(right),
                max(right) - min(left))
```

Analyzing the Algorithm

```
procedure maxProfit(list prices):
  if length(prices) ≤ 1:
     return 0

  let left  be the first half of prices
  let right be the second half of prices

  return max(maxProfit(left), maxProfit(right),
             max(right) - min(left))
```

$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n)$$

$$\mathbf{T(\boldsymbol{n}) = O(\boldsymbol{n} \log \boldsymbol{n})}$$

The Divide-and-Conquer Framework

- The two algorithms we have just seen are examples of **divide-and-conquer** algorithms.
- These algorithms usually have two steps:
  - **(Divide)** Split the input apart into multiple smaller pieces, recursively solving each piece.
  - **(Conquer)** Combine the solutions to each smaller piece together into the overall solution.
- Typically, correctness is proven inductively and runtime is proven by solving a recurrence relation.
- In many cases, the runtime is determined without actually solving the recurrence; more on that later.

Another Algorithm: **Binary Search**

```
procedure binarySearch(list A, int low, int high,
                               value key):
  if low ≥ high:
     return false

  let mid = ⌊(high + low) / 2⌋
  if A[mid] = key:
     return true
  else if A[mid] > key:
     return binarySearch(a, low, mid)
  else (A[mid] < key):
     return binarySearch(a, mid + 1, high)
```

$$T(1) \leq c$$
$$T(n) \leq T(n / 2) + c$$

The Iteration Method

T(1) ≤ c
T(n) ≤ T(n / 2) + c

$$
\begin{aligned}
\mathrm{T}(n) &\le T\left(\frac{n}{2}\right)+c \\
&\le \left(T\left(\frac{n}{4}\right)+c\right)+c \\
&= T\left(\frac{n}{4}\right)+2c \\
&\le \left(T\left(\frac{n}{8}\right)+c\right)+2c \\
&= T\left(\frac{n}{8}\right)+3c \\
&\ldots \\
&\le T\left(\frac{n}{2^k}\right)+kc
\end{aligned}
$$

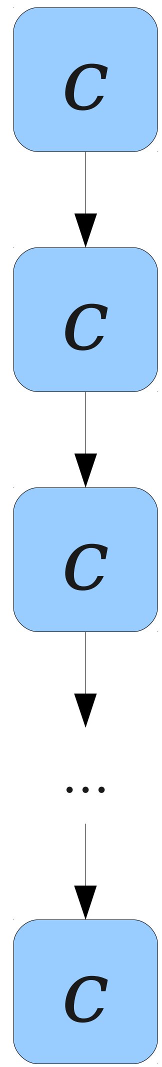
The Iteration Method

$$\boxed{\begin{array}{l} \mathrm{T}(1) \le c \\ \mathrm{T}(n) \le \mathrm{T}(n / 2) + c \end{array}}$$

$$\begin{aligned} \mathrm{T}(n) &\le T\left(\frac{n}{2^k}\right) + k\,c \\ &= T(1) + c\log_2 n \\ &\le c + c\log_2 n \\ &= O(\log n) \end{aligned}$$

The Recursion Tree Method

$T(1) \leq c$
$T(n) \leq T(n / 2) + c$



$c \log_2 n + c$

Formalizing Our Argument

- To formalize correctness, it's useful to use this invariant:

If $key = A[i]$ for some i, then
$low \leq i < high$

- You can prove this is true by induction on the number of calls made.
- We can also formalize the runtime bound by induction to prove the O(log n) upper bound, but it's not super exciting to do so.