Chapter

3

The variational formulation of elliptic PDEs

We now begin the theoretical study of elliptic partial differential equations and boundary value problems. We will focus on one approach, which is called the variational approach. There are other ways of solving elliptic problems. The variational approach is quite simple and well suited for a whole class of approximation methods, as we will see later.

3.1 Model problems

Let us start with a few model problems. The simplest of all is a slight generalization of the Poisson equation with a homogeneous Dirichlet boundary condition. Let us be given a connected[1] open Lipschitz subset Ω of $\mathbb{R}^d$, a function $c \in L^\infty(\Omega)$, another function $f \in L^2(\Omega)$. We are looking for a function $u\colon \bar{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u + cu = f \text{ in } \Omega, \\ \qquad\quad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

We are going to transform the boundary value problem (3.1) into an entirely different kind of problem that is amenable to an existence and uniqueness theory, as well as the definition of approximation methods.

[1]In the context of PDEs, all the open subsets considered will be connected, without further mention. It is not that important, but PDEs set on open sets with several connected components are basically several unrelated PDEs.

Proposition 3.1.1 *Assume that $u \in H^2(\Omega)$ solves the PDE in problem* (3.1). *Then, for all $v \in H_0^1(\Omega)$*

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega cuv \, dx = \int_\Omega fv \, dx. \tag{3.2}$$

Proof. We take an arbitrary $v \in H_0^1(\Omega)$, multiply the equation by v,

$$-(\Delta u)v + cuv = fv,$$

and integrate over Ω. Every term is integrable since $u \in H^2(\Omega)$ hence $\Delta u \in L^2(\Omega)$ and $v \in L^2(\Omega)$ imply $(\Delta u)v \in L^1(\Omega)$, $c \in L^\infty(\Omega)$, $u \in L^2(\Omega)$ and $v \in L^2(\Omega)$ imply $cuv \in L^1(\Omega)$, and $f \in L^2(\Omega)$ implies $fv \in L^1(\Omega)$. We thus obtain

$$-\int_\Omega (\Delta u) v \, dx + \int_\Omega cuv \, dx = \int_\Omega fv \, dx.$$

We now use Green's formula (2.17), according to which

$$\int_\Omega (\Delta u) v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \gamma_1(u)\gamma_0(v) \, d\Gamma,$$

and we conclude since $v \in H_0^1(\Omega)$ is equivalent to $\gamma_0(v) = 0$. □

Formulation (3.2) is called the *variational formulation* of problem (3.1). Actually, it is not entirely complete, since we have not yet decided in which space to look for u. In fact, as we have seen in the previous section, the reasonable way to impose the Dirichlet boundary condition is to require that $u \in H_0^1(\Omega)$. The functions v are called *test-functions*.

Let us rewrite the variational formulation in a standard, abstract form. We let $V = H_0^1(\Omega)$, it is a Hilbert space. Then we have a bilinear form on $V \times V$

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + cuv) \, dx$$

and a linear form on V

$$\ell(v) = \int_\Omega fv \, dx.$$

The variational formulation then reads

$$\forall v \in V, \quad a(u,v) = \ell(v), \tag{3.3}$$

and we have shown that a solution of the boundary value problem with the additional regularity $u \in H^2(\Omega)$ is a solution of the variational problem (3.3).

Now what about the reverse implication? Does a solution of the variational problem solve the boundary value problem? The answer is basically yes, the two problems are equivalent.

Proposition 3.1.2 *Assume that $u \in V$ solves the variational problem* (3.3). *Then we have*

$$-\Delta u + cu = f \text{ in the sense of } \mathscr{D}'(\Omega)$$

and $\Delta u \in L^2(\Omega)$.

Proof. We have $\mathscr{D}(\Omega) \subset H_0^1(\Omega)$, therefore we can take $v = \varphi \in \mathscr{D}(\Omega)$ as test-function in (3.3). Let us examine each term separately.

First of all

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \sum_{i=1}^d \partial_i u \partial_i \varphi \, dx = \sum_{i=1}^d \langle \partial_i u, \partial_i \varphi \rangle = \sum_{i=1}^d -\langle \partial_{ii} u, \varphi \rangle = -\langle \Delta u, \varphi \rangle,$$

by definition of distributional derivatives. Similarly

$$\int_\Omega cu\varphi \, dx = \langle cu, \varphi \rangle \text{ and } \int_\Omega f\varphi \, dx = \langle f, \varphi \rangle.$$

Therefore, we have for all $\varphi \in \mathscr{D}(\Omega)$

$$\langle -\Delta u + cu - f, \varphi \rangle = 0$$

or

$$-\Delta u + cu - f = 0 \text{ in the sense of } \mathscr{D}'(\Omega)$$

and the PDE is satisfied in the sense of distributions The Dirichlet boundary condition is also satisfied by the simple fact that $u \in H_0^1(\Omega)$, hence the boundary value problem is solved.

To conclude, we note that $\Delta u = cu - f \in L^2(\Omega)$. This also implies that the PDE is satisfied almost everywhere. □

Remark 3.1.1 The two problems are thus equivalent, except for the fact that we have assumed $u \in H^2(\Omega)$ in one direction, and only recuperated $\Delta u \in L^2(\Omega)$ in the other.[2] Actually, the assumption $u \in H^2(\Omega)$ is somewhat artificial and made only to make use of Green's formula (2.17). It is possible to dispense with it with a little more work, but that would take us too far.

It should be noted in any case, that if $u \in H_0^1(\Omega)$, $\Delta u \in L^2(\Omega)$ and Ω is for example of class C^2, then $u \in H^2(\Omega)$. This is very profound result in *elliptic regularity theory*, far beyond the scope of these notes. It is trivial in dimension one though.

Of course, so far we have no indication that either problem has a solution. The fact is that the variational formulation is significantly easier to treat, once the right point of view is found. And the right point of view is an abstract point of view, as is often the case. □

[2]The Laplacian is a specific linear combination of some of the second order derivatives. So it being in L^2 is a priori less than all individual second order derivatives, even those not appearing in the Laplacian, being in L^2.

Before we start delving in the abstract, let us give a couple more model problems of a different kind. First is a new boundary condition.

$$\begin{cases} -\Delta u + cu = f \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega. \end{cases} \tag{3.4}$$

This is an example of a *Neumann boundary condition*. When $g = 0$, it is naturally called a homogeneous Neumann boundary condition. In terms of modeling, the Neumann condition is a flux condition. For instance, in the heat equilibrium interpretation, the condition corresponds to an imposed heat flux through the boundary, as opposed to the Dirichlet condition which imposes a temperature on the boundary. The case $g = 0$ corresponds to perfect thermal insulation.

Let us derive the variational formulation informally. Assume first that $u \in H^2$, take $v \in H^1(\Omega)$, multiply, integrate and use Green's formula to obtain

$$\forall v \in H^1(\Omega), \quad \int_\Omega (\nabla u \cdot \nabla v + cuv)\, dx = \int_\Omega fv\, dx + \int_{\partial\Omega} g\gamma_0(v)\, d\Gamma.$$

Note the different test-function space and the additional boundary term in the right-hand side.

The converse is more interesting. Let $u \in H^2(\Omega)$ be a solution of the above variational problem. Taking first $v = \varphi \in \mathscr{D}(\Omega)$, we obtain

$$-\Delta u + cu = f \text{ in the sense of } \mathscr{D}'(\Omega)$$

exactly as in the Dirichlet case. Of course, a test-function with compact support does not see what happens on the boundary, and no information on the Neumann condition is recovered. Thus, in a second step, we take v arbitrary in $H^1(\Omega)$. By Green's formula again, we have

$$\int_\Omega \nabla u \cdot \nabla v\, dx = -\int_\Omega (\Delta u) v\, dx + \int_{\partial\Omega} \gamma_1(u)\gamma_0(v)\, d\Gamma.$$

Recall that the normal trace $\gamma_1(u)$ plays the role of the normal derivative. Since u is a solution of the variational problem, it follows that

$$\int_\Omega (-\Delta u + cu) v\, dx + \int_{\partial\Omega} \gamma_1(u)\gamma_0(v)\, d\Gamma = \int_\Omega fv\, dx + \int_{\partial\Omega} g\gamma_0(v)\, d\Gamma.$$

But we already know that $\int_\Omega (-\Delta u + cu) v\, dx = \int_\Omega fv\, dx$, hence we are left with

$$\int_{\partial\Omega} \gamma_1(u)\gamma_0(v)\, d\Gamma = \int_{\partial\Omega} g\gamma_0(v)\, d\Gamma,$$

for all $v \in H^1(\Omega)$. For simplicity, we assume here that $g \in H^{1/2}(\partial\Omega)$, the image of the trace γ_0 and that Ω is smooth. Since $u \in H^2(\Omega)$, it follows that $\gamma_1(u) =$

$\sum_{i=1}^{d} \gamma_0(\partial_i u) n_i \in H^{1/2}(\partial\Omega)$. Therefore, there exists $v \in H^1(\Omega)$ such that $\gamma_0(v) = \gamma_1(u) - g$. With this choice of v, we obtain

$$\int_{\partial\Omega} (\gamma_1(u) - g)^2 \, d\Gamma = 0,$$

hence $\gamma_1(u) = g$ which is the Neumann condition. □

The last hypotheses made are for brevity only. They are not at all necessary to conclude. Another problem of interest is the non homogeneous Dirichlet problem.

$$\begin{cases} -\Delta u + cu = f \text{ in } \Omega, \\ \qquad\quad u = g \text{ on } \partial\Omega, \end{cases} \tag{3.5}$$

with $g \in H^{1/2}(\partial\Omega)$. This problem is reduced to the homogeneous problem by taking a function $G \in H^1(\Omega)$ such that $\gamma_0(G) = g$ and setting $U = u - G$. Then clearly $U \in H_0^1(\Omega)$ and $-\Delta U + cU = -\Delta u + cu + \Delta G - cG = f + \Delta G - cG$. Then we just write the variational formulation of the homogeneous problem for U with right-hand side $F = f + \Delta G - cG$.

The Dirichlet and Neumann conditions can be mixed together, but *not at the same place* on the boundary, yielding the so-called *mixed problem*. More precisely, let Γ_1 and Γ_2 be two subsets of $\partial\Omega$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \partial\Omega$. Then the problem

$$\begin{cases} -\Delta u + cu = f \text{ in } \Omega, \\ \qquad\quad u = g_1 \text{ on } \Gamma_1, \\ \qquad\quad \frac{\partial u}{\partial n} = g_2 \text{ on } \Gamma_2. \end{cases} \tag{3.6}$$

The variational formulation for the mixed problem (in the case $g_1 = 0$ for brevity, if not follow the above route) is to let $V = \{v \in H^1(\Omega); \gamma_0(v) = 0 \text{ on } \Gamma_1\}$ and

$$\forall v \in V, \quad \int_\Omega (\nabla u \cdot \nabla v + cuv) \, dx = \int_\Omega f v \, dx + \int_{\Gamma_2} g_2 \gamma_0(v) \, d\Gamma,$$

with $u \in V$.

Remark 3.1.2 An important rule of thumb to be remembered from the above examples is that (homogeneous) Dirichlet conditions are taken into account in the test-function space, whereas Neumann boundary conditions are taken into account in the linear form via boundary integrals.

3.2 Abstract variational problems

We describe the general abstract framework for all variational problems. Let us start with a quick review of Hilbert space theory. Let H be a real Hilbert space with scalar product $(\cdot|\cdot)_H$ and norm $\|\cdot\|_H$. We first recall the Cauchy-Schwarz inequality, which is really a hilbertian property.

Theorem 3.2.1 *For all $u, v \in H$, we have*

$$|(u,v)_H| \leq \|u\|_H \|v\|_H.$$

One of the most basic results in Hilbert space theory is the orthogonal projection theorem, which we recall below.

Theorem 3.2.2 *Let C be non empty, convex, closed subset of H. For all $x \in H$, there exists a unique $p_C(x) \in C$ such that*

$$\|x - p_C(x)\|_H = \inf_{y \in C} \|x - y\|_H.$$

The vector $p_C(x)$ is called the orthogonal projection of x on C. *It is also characterized by the inequality*

$$\forall y \in C, \quad (x - p_C(x) | y - p_C(x))_H \leq 0.$$

In addition, if C is a closed vector subspace of H, then p_C is a continuous linear mapping from H to C which is also characterized by the equality

$$\forall y \in C, \quad (x - p_C(x) | y)_H = 0.$$

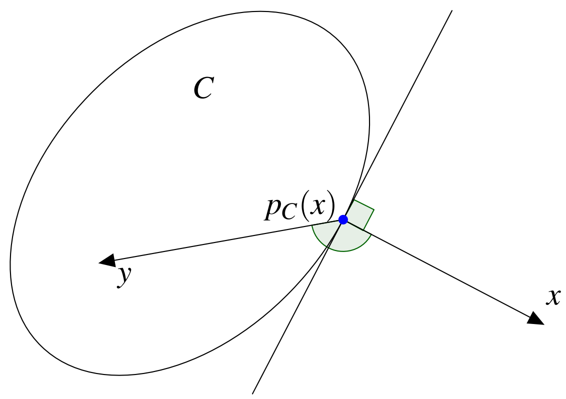

Figure 1. The orthogonal projection on a closed convex C.

The orthogonal projection of x on C is thus the element of C closest to x and the angle between $x - p_C(x)$ and $y - p_C(x)$ is larger than $\frac{\pi}{2}$. In particular, if $x \in C$, then $p_C(x) = x$. An important consequence of the last characterization in the case of a closed vector subspace is that we can write $H = C \oplus C^\perp$ with continuous orthogonal projections on each factor. Indeed, we have $x = p_C(x) + (x - p_C(x)$ with $p_C(x) \in C$ by construction and $x - p_C(x) \in C^\perp$ by the second characterization. Hence $H = C + C^\perp$. To show that the sum is direct, it suffices to note that $C \cap C^\perp = \{0\}$ which is obvious since $x \in C \cap C^\perp$ implies $0 = (x|x)_H = \|x\|_H^2$.

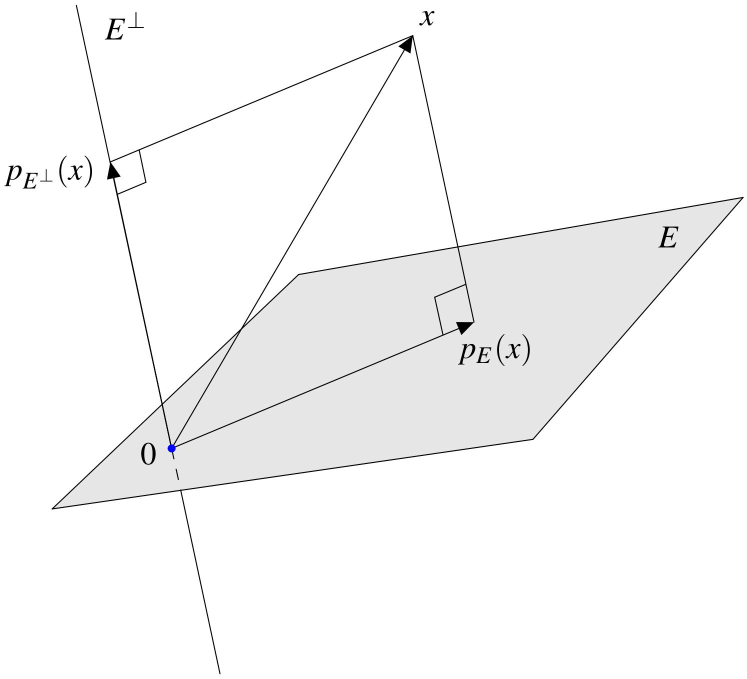

Figure 2. The orthogonal projection on a closed vector subspace E.

Another important consequence is a characterization of dense subspaces.

Lemma 3.2.1 *A vector subspace E of H is dense in H if and only of $E^{\perp} = \{0\}$.*

Proof. For any vector subspace, it is always true that $E^{\perp} = (\bar{E})^{\perp}$. Let E be a dense subspace, *i.e.*, $\bar{E} = H$. Then, of course $E^{\perp} = H^{\perp} = \{0\}$. Conversely, if $E^{\perp} = \{0\}$, this implies that $(\bar{E})^{\perp} = \{0\}$ and since $H = (\bar{E})^{\perp} \oplus \bar{E}$, it follows that $\bar{E} = H$, and E is dense in H. □

The Riesz theorem provides a canonical way of identifying a Hilbert space and its dual.

Theorem 3.2.3 (Riesz) *Let H be a Hilbert space and ℓ an element of its dual H'. There exists a unique $u \in H$ such that*

$$\forall v \in H, \quad \ell(v) = (u|v)_H.$$

Moreover

$$\|\ell\|_{H'} = \|u\|_H$$

and the linear mapping $\delta \colon H' \to H$, $\ell \mapsto u$, is an isometry.

Proof. If $\ell = 0$, then we set $u = 0$ to be the unique u in question. Let $\ell \neq 0$. It is thus a nonzero continuous linear form, hence its kernel $\ker \ell$ is a closed hyperplane of H. Let us choose $u_0 \in (\ker \ell)^\perp$ with $\|u_0\|_H = 1$ (this is possible since $\ker \ell$ is not dense). Since $u_0 \notin \ker \ell$, we have $\ell(u_0) \neq 0$ and for all $v \in H$, we can set $w = v - \frac{\ell(v)}{\ell(u_0)} u_0$ so that

$$\ell(w) = \ell\Big(v - \frac{\ell(v)}{\ell(u_0)} u_0\Big) = \ell(v) - \frac{\ell(v)}{\ell(u_0)} \ell(u_0) = 0,$$

and $w \in \ker \ell$. Now writing $v = \frac{\ell(v)}{\ell(u_0)} u_0 + w$ and setting $u = \ell(u_0) u_0 \in (\ker \ell)^\perp$, we obtain

$$(v|u)_H = \Big(\frac{\ell(v)}{\ell(u_0)} u_0 \Big| u\Big)_H + (w|u)_H = \ell(v)(u_0|u_0)_H = \ell(v),$$

hence the existence of u.

For the uniqueness, assume u_1 and u_2 are two solutions, then for all $v \in H$, $(v|u_1 - u_2)_H = 0$. This is true in particular for $v = u_1 - u_2$, hence $u_1 = u_2$.

The mapping δ is thus well defined and obviously linear. Finally, for the isometry, we have on the one hand

$$\|\ell\|_{H'} = \sup_{\|v\|_H \leq 1} |\ell(v)| = \sup_{\|v\|_H \leq 1} |(v|u)_H| \leq \|v\|_H \|u\|_H \leq \|u\|_H,$$

by the Cauchy-Schwarz inequality. On the other hand, equality trivially holds for $\ell = 0$, and for $\ell \neq 0$, choosing $v = \frac{u}{\|u\|_H}$ yields $(v|u)_H = \|u\|_H$ with $\|v\|_H = 1$, hence the equality in this case too. □

Remark 3.2.1 The Riesz theorem shows that the dual of a Hilbert space is also a Hilbert space for the scalar product $(\ell_1|\ell_2)_{H'} = (\delta\ell_1|\delta\ell_2)_H$, since it is not obvious a priori that the dual norm is hilbertian. It is often used to identify H and H' via the isometry δ or δ^{-1}. This identification is not systematic however. For example, when we have two Hilbert spaces H and V such that $V \hookrightarrow H$ and V is dense in H, the usual identification is to let

$$V \hookrightarrow H = H' \hookrightarrow V'$$

using the Riesz theorem for H, which is called the pivot space, but not for V. Such is the case for $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ in which case we have an identification of V' as the space $H^{-1}(\Omega)$. Indeed, the scalar product used in the identification of the pivot space and its dual is the duality bracket of an L^2 function seen as a distribution and a $\mathscr{D}$ test-function. This is not the case for any of the two scalar products that H_0^1 comes equipped with, and an identification using these scalar products, which is also legitimate, does not yield a space of distributions. □

We now come to abstract variational problems and how they are solved. This is the Lax-Milgram theorem. This theorem is important, not because it is in any way difficult, which it is not, but because it has a very wide range of applicability as we will see later.

Theorem 3.2.4 (Lax-Milgram) *Let V be a Hilbert space, a be a bilinear form and ℓ be a linear form. Assume that*

i) *The bilinear form a is continuous,* i.e.*, there exists a constant M such that $|a(u,v)| \leq M\|u\|_V\|v\|_V$ for all $u,v \in V$,*

ii) *The bilinear form a is V*-elliptic[3], i.e.*, there exists a constant $\alpha > 0$ such that $a(v,v) \geq \alpha\|v\|_V^2$ for all $v \in V$,*

iii) *The linear form ℓ is continuous,* i.e.*, there exists a constant C such that $|\ell(v)| \leq C\|v\|_V$ for all $v \in V$.*

There exists a unique $u \in V$ that solves the abstract variational problem: Find $u \in V$ such that

$$\forall v \in V, \quad a(u,v) = \ell(v). \tag{3.7}$$

Proof. Let us start with the uniqueness. Let u_1 and u_2 be two solutions of problem (3.7). Since a is linear with respect to its first argument, it follows that $a(u_1 - u_2, v) = 0$ for all $v \in V$. In particular, for $v = u_1 - u_2$, we obtain

$$0 = a(u_1 - u_2, u_1 - u_2) \geq \alpha\|u_1 - u_2\|_V^2,$$

so that $\|u_1 - u_2\|_V = 0$ since $\alpha > 0$.

We next prove the existence of a solution. We first note that for all $u \in V$, the mapping $v \mapsto a(u,v)$ is linear (by bilinearity of a) and continuous (by i) continuity of a). Therefore, there exists a unique element Au of V' such that $a(u,v) = \langle Au, v\rangle_{V',V}$. Moreover, the bilinearity of a shows that the mapping $A\colon V \to V'$ thus defined is linear. It is also continuous since for all $v \in V$, $\|v\|_V \leq 1$,

$$|\langle Au, v\rangle_{V',V}| = |a(u,v)| \leq M\|u\|_V\|v\|_V \leq M\|u\|_V$$

so that

$$\|Au\|_{V'} = \sup_{\|v\|_V \leq 1} |\langle Au, v\rangle_{V',V}| \leq M\|u\|_V.$$

We rewrite the variational problem as: Find $u \in V$ such that

$$\forall v \in V, \quad \langle Au - \ell, v\rangle_{V',V} = 0$$

or

$$Au = \ell,$$

[3] This condition is also sometimes called coerciveness.

and this is where the continuity of ℓ is used

Thus, proving the existence is equivalent to showing that the mapping A is onto. We do this in two independent steps: we show that $\operatorname{im} A$ is closed on the one hand and that it is dense on the other hand.[4] For the closedness of the image, we use assuption ii) of V-ellipticity. Let ℓ_n be a sequence in $\operatorname{im} A$ such that $\ell_n \to \ell$ in V'. We want to show that $\ell \in \operatorname{im} A$, which will imply that $\operatorname{im} A$ is closed. The sequence ℓ_n is a Cauchy sequence in V', and for all n, there exists $u_n \in V$ such that $Au_n = \ell_n$. By V-ellipticity,

$$\begin{aligned}\|u_n - u_m\|_V^2 &\leq \frac{1}{\alpha} a(u_n - u_m, u_n - u_m) = \frac{1}{\alpha}\langle Au_n - Au_m, u_n - u_m\rangle_{V',V} \\ &= \frac{1}{\alpha}\langle \ell_n - \ell_m, u_n - u_m\rangle_{V',V} \leq \frac{1}{\alpha}\|\ell_n - \ell_m\|_{V'}\|u_n - u_m\|_V,\end{aligned}$$

by the definition of the dual norm. Therefore, if $\|u_n - u_m\|_V = 0$ we are happy, otherwise we divide by $\|u_n - u_m\|_V$ and in both cases

$$\|u_n - u_m\|_V \leq \frac{1}{\alpha}\|\ell_n - \ell_m\|_{V'},$$

so that u_n is a Cauchy sequence in V. Since V is complete, there exists $u \in V$ such that $u_n \to u$ in V. Since A is continuous, it follows that $\ell_n = Au_n \to Au$ in V'. Hence $\ell = Au \in \operatorname{im} A$.

To show the density, we show that $(\operatorname{im} A)^\perp = \{0\}$. We note that $(Au|\ell)_{V'} = (\delta Au|\delta\ell)_V = \langle Au, \delta\ell\rangle_{V',V} = a(u, \delta\ell)$. Let $\ell \in (\operatorname{im} A)^\perp$. For all $u \in V$, we thus have $a(u, \delta\ell) = 0$. In particular, for $u = \delta\ell$, we obtain $0 = a(\delta\ell, \delta\ell) \geq \alpha\|\delta\ell\|_V^2$ by V-ellipticity. Since $\alpha > 0$, it follows that $\ell = 0$. □

Remark 3.2.2 It should be noted that the Lax-Milgram theorem is not a particular case of the Riesz theorem. It is actually more general, since it applies to bilinear forms that are not necessarily symmetric, and it implies the Riesz theorem when the bilinear form is just the scalar product.

Sometimes when the bilinear form a is symmetric, people think it advantageous to apply Riesz's theorem in place of the Lax-Milgram theorem. This is usually an illusion: indeed, if a new scalar product defined by the bilinear form is introduced, in order to apply Riesz's theorem, it is necessary to show that the space equipped with the new scalar product is still a Hilbert space, *i.e.*, is complete. This is done by V-ellipticity, hence nothing is gained (although this is the part that people who think they are seeing a good deal usually forget). The continuity of the linear form for the new norm must also be checked, which amounts to having the bilinear form and linear form continuous for the original norm, again, no gain.

[4] This is a pretty common strategy, to be kept in mind.

The only case when Riesz's theorem can be deemed advantageous over the Lax-Milgram theorem, is when both above facts to be checked are already known. An example is the bilinear form $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$ on $V = H_0^1(\Omega)$. □

Remark 3.2.3 In the case of complex Hilbert spaces and complex-valued variational problems, the Lax-Milgram theorem still holds true for a bilinear or sesquilinear form. The V-ellipticity assumption can even be relaxed to only involve the real part of $a : \Re(a(u,u)) \geq \alpha \|u\|^2$ (or the imaginary part), which is rather useful as the imaginary part can then be pretty arbitrary (exercise). □

The linear form in the right-hand side of a variational problem should be thought of as data. In this respect, the solution depends continuously on the data.

Proposition 3.2.1 *The mapping $V' \to V$, $\ell \mapsto u$ defined by the Lax-Milgram theorem is linear and continuous.*

Proof. Let ℓ_1, ℓ_2 be two linear forms and u_1, u_2 the corresponding solutions to the variational problem. For all $\lambda \in \mathbb{R}$, we have for all $v \in V$,

$$a(u_1 + \lambda u_2, v) = a(u_1, v) + \lambda a(u_2, v) = \ell_1(v) + \lambda \ell_2(v) = (\ell_1 + \lambda \ell_2)(v)$$

hence the linearity by uniqueness of the solution. For the continuity, we have

$$\alpha \|u\|_V^2 \leq a(u,u) = \ell(u) \leq \|\ell\|_{V'} \|u\|_V,$$

hence

$$\|u\|_V \leq \frac{1}{\alpha} \|\ell\|_{V'},$$

and we have the continuity, with continuity constant $\frac{1}{\alpha}$. □

Proposition 3.2.2 *Let the hypotheses of the Lax-Milgram theorem be satisfied. Assume in addition that the bilinear form a is symmetric. Then the solution u of the variational problem* (3.7) *is also the unique solution of the minimization problem:*

$$J(u) = \inf_{v \in V} J(v) \quad \textit{with} \quad J(v) = \frac{1}{2} a(v,v) - \ell(v). \tag{3.8}$$

Proof. Let u be the Lax-Milgram solution. For all $v \in V$, we let $w = v - u$ and

$$\begin{aligned} J(v) = J(u+w) &= \frac{1}{2} a(u,u) + \frac{1}{2} a(u,w) + \frac{1}{2} a(w,u) + \frac{1}{2} a(w,w) - \ell(u) - \ell(w) \\ &= J(u) + a(u,w) - \ell(w) + \frac{1}{2} a(w,w) \\ &\geq J(u), \end{aligned}$$

since $a(w,w) \geq 0$. Make note of where the symmetry is used. Hence, u minimizes J on V.

Conversely, assume that u minimizes J on V. Then, for all $\lambda > 0$ and all $v \in V$, we have $J(u+\lambda v) \geq J(u)$. Expanding the left-hand side, we get

$$\frac{1}{2}a(u,u) + \lambda a(u,v) + \frac{\lambda^2}{2}a(v,v) - \ell(u) - \lambda \ell(v) \geq J(u)$$

so that dividing by λ

$$a(u,v) - l(v) + \frac{\lambda}{2}a(v,v) \geq 0.$$

We then let $\lambda \to 0$, hence

$$a(u,v) - l(v) \geq 0,$$

and finally change v in $-v$ to obtain

$$a(u,v) - l(v) = 0,$$

for all $v \in V$. □

Remark 3.2.4 Taking $\lambda > 0$, dividing by λ and then letting $\lambda \to 0$ is quite clever, and known as Minty's trick. □

Remark 3.2.5 When the bilinear form a is not symmetric, we can still define the functional J in the same fashion as before and try to minimize it. It is clear from the above proof that the minimizing element u does not solve the variational problem associated with a but the variational problem associated with the symmetric part of a. Of course, when both variational problems are translated into PDEs, we get entirely different equations. □

3.3 Application to the model problems, and more

We now apply the abstract results to concrete examples. We start with the first model problem (3.1).

Proposition 3.3.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$, $f \in L^2(\Omega)$, $c \in L^\infty(\Omega)$. Assume that $c \geq 0$. Then the problem: Find $u \in V = H_0^1(\Omega)$ such that*

$$\forall v \in V, \quad \int_\Omega (\nabla u \cdot \nabla v + cuv)\,dx = \int_\Omega fv\,dx,$$

has one and only one solution.

Proof. We already that V is a Hilbert space, for both scalar products that we defined earlier. Of course

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + cuv)\,dx$$

clearly defines a bilinear form on $V \times V$ and

$$\ell(v) = \int_\Omega fv\,dx$$

a linear form on V. Hence, we have an abstract variational problem. Let us try and apply the Lax-Milgram theorem. We need to check the hypotheses. For definiteness, we choose to work with the full H^1 norm.

First of all, for all $(u,v) \in V \times V$,

$$\begin{aligned}
|a(u,v)| &= \left| \int_\Omega (\nabla u \cdot \nabla v + cuv)\,dx \right| \\
&\leq \int_\Omega |\nabla u \cdot \nabla v + cuv|\,dx \\
&\leq \int_\Omega |\nabla u \cdot \nabla v|\,dx + \int_\Omega |cuv|\,dx \\
&\leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq \max\big(1, \|c\|_{L^\infty(\Omega)}\big) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)},
\end{aligned}$$

by Cauchy-Schwarz to go from the third line to the fourth line, hence the continuity of the bilinear form a.

Next is the V-ellipticity. For all $v \in V$, we have

$$a(v,v) = \int_\Omega (\|\nabla v\|^2 + cv^2)\,dx \geq \int_\Omega \|\nabla v\|^2\,dx \geq \alpha \|v\|^2_{H^1(\Omega)}$$

with $\alpha = (C^2+1)^{-1/2} > 0$ by Corollary 2.6.3 where C is the Poincaré inequality constant, and since $c \geq 0$.

Finally, we check the continuity of the linear form. For all $v \in V$,

$$|\ell(v)| = \left| \int_\Omega fv\,dx \right| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}$$

by Cauchy-Schwarz again.

All the hypotheses of the Lax-Milgram theorem are satisfied, therefore there is one and only one solution $u \in V$. □

Remark 3.3.1 Now is a time to celebrate since we have successfully solved our first boundary value problem in arbitrary dimension. Indeed, we have already seen that any solution of the variational problem is a solution of the PDE in the distributional sense and in the L^2 sense. The solution u depends continuously in H^1 on f in L^2. Note that we have also solved the non homogeneous Dirichlet problem at the same time. It is an instructive exercise to redo the proof using the H^1 semi-norm in place of the full norm. The same ingredients are used, but not at the same spots.

This is a case of a symmetric bilinear form, therefore the solution u also minimizes the so-called energy functional

$$J(v) = \frac{1}{2}\int_\Omega (\|\nabla v\|^2 + cv^2)\,dx - \int_\Omega fv\,dx$$

over V. □

Remark 3.3.2 It should be noted that the positivity condition $c \geq 0$ is by no means a necessary condition for existence and uniqueness via the Lax-Milgram theorem. With a little more work, it is not too hard to allow the function c to take some negative values. However, we have seen an example at the very beginning of these notes with a negative function c for which existence and uniqueness fails.

One should also be aware that there is an existence and uniqueness theory that goes beyond the Lax-Milgram theorem, which is only a sufficient condition for existence and uniqueness. □

Let us now consider the non homogeneous Neumann problem (3.4). The hypotheses are slightly different.

Proposition 3.3.2 *Let Ω be a bounded open subset of $\mathbb{R}^n$, $f \in L^2(\Omega)$, $c \in L^\infty(\Omega)$, $g \in L^2(\partial\Omega)$. Assume that there exists a constant $\eta > 0$ such that $c \geq \eta$ almost everywhere. Then the problem: Find $u \in V = H^1(\Omega)$ such that*

$$\forall v \in V, \quad \int_\Omega (\nabla u \cdot \nabla v + cuv)\,dx = \int_\Omega fv\,dx + \int_{\partial\Omega} g\gamma_0(v)\,dx,$$

has one and only one solution.

Proof. We have a different Hilbert space (but known to be Hilbert, nothing to check here), the same bilinear form and a different linear form

$$\ell(v) = \int_\Omega fv\,dx + \int_{\partial\Omega} g\gamma_0(v)\,dx.$$

We have already shown that the bilinear form is continuous in the H^1 norm[5]. The V-ellipticity is clear since, for all $v \in V$,

$$a(v,v) = \int_\Omega (\|\nabla v\|^2 + cv^2)\,dx \geq \int_\Omega \|\nabla v\|^2\,dx + \eta \int_\Omega v^2\,dx \geq \min(1,\eta)\|v\|^2_{H^1(\Omega)},$$

with $\min(1,\eta) > 0$. The continuity of the linear form is also clear

$$\begin{aligned} |\ell(v)| &\leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\|\gamma_0(v)\|_{L^2(\partial\Omega)} \\ &\leq (\|f\|_{L^2(\Omega)} + C\|g\|_{L^2(\partial\Omega)})\|v\|_{H^1(\Omega)} \end{aligned}$$

by Cauchy-Schwarz, where C is continuity constant of the trace mapping. □

The mixed problem (3.6) is practically entirely identical to the Neumann problem.

Proposition 3.3.3 *Same hypotheses as in Proposition* 3.3.2 *and let* Γ_1 *and* Γ_2 *be two subsets of* $\partial\Omega$ *such that* $\Gamma_1 \cap \Gamma_2 = \emptyset$ *and* $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \partial\Omega$. *Then the problem: Find* $u \in V = \{v \in H^1(\Omega); \gamma_0(v) = 0 \text{ on } \Gamma_1\}$ *such that*

$$\forall v \in V, \quad \int_\Omega (\nabla u \cdot \nabla v + cuv)\,dx = \int_\Omega fv\,dx + \int_{\Gamma_2} g\gamma_0(v)\,d\Gamma,$$

has one and only one solution.

Proof. The only real difference with Proposition 3.3.2 lies with the space V, which we do not know yet to be a Hilbert space. It suffices to show that V is a closed subspace of $H^1(\Omega)$. Let v_n be a sequence in V such that $v_n \to v$ in $H^1(\Omega)$. By continuity of the trace mapping, we have $\gamma_0(v_n) \to \gamma_0(v)$ in $L^2(\partial\Omega)$. Therefore, there exists a subsequence $\gamma_0(v_{n_p})$ that converges to $\gamma_0(v)$ almost everywhere on $\partial\Omega$. Since $\gamma_0(v_n) = 0$ almost everywhere on Γ_1, it follows that $\gamma_0(v) = 0$ almost everywhere on Γ_1, hence $v \in V$, which is thus closed. □

A natural question arises about the Neumann problem without a strictly positive bound from below for the function c, in particular for $c = 0$. Now, this is an entirely different problem from the previous ones. First we have to find the variational formulation of the boundary value problem and show that it is equivalent to the boundary value problem, then we have to apply the Lax-Milgram theorem.

Let us thus consider the Neumann problem

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega, \end{cases} \tag{3.9}$$

[5] If we had worked with the semi-norm for the Dirichlet problem, we would have had to do the continuity all over again here...

in a Lipschitz open set Ω in $\mathbb{R}^d$. We see right away that things are going to be different since we do not have uniqueness here. Indeed, if u is a solution, then $u+s$ is also a solution for any constant s. Furthermore, by Green's formula (2.17) with $v=1$, it follows that if there is a solution, then, necessarily

$$\int_\Omega f\,dx + \int_{\partial\Omega} g\,d\Gamma = 0. \tag{3.10}$$

If the data f,g does not satisfy the compatibility condition (3.10), there is thus no solution. The two remarks, non uniqueness and non existence, are actually dual to each other.

There are several ways of going around both problems, thus several variational formulations[6]. We choose to set,

$$V = \Big\{ v \in H^1(\Omega); \int_\Omega v\,dx = 0 \Big\}.$$

This is well defined, since Ω is bounded and we thus have $H^1(\Omega) \subset L^2(\Omega) \subset L^1(\Omega)$. Note that V is the L^2-orthogonal to the one-dimensional space of constant functions, which are the cause of non uniqueness.

Lemma 3.3.1 *The space V is a Hilbert space for the scalar product of $H^1(\Omega)$.*

Proof. It suffices to show that V is closed. Let v_n be a sequence in V such that $v_n \to v$ in $H^1(\Omega)$. Of course, $v_n \to v$ in $L^2(\Omega)$ and by Cauchy-Schwarz, $v_n \to v$ in $L^1(\Omega)$. Therefore

$$0 = \int_\Omega v_n\,dx \to \int_\Omega v\,dx,$$

and $v \in V$. □

Proposition 3.3.4 *Assume that $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ satisfy the compatibility condition* (3.10). *Then the triple V, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx$, $\ell(v) = \int_\Omega f v\,dx + \int_{\partial\Omega} g\gamma_0(v)\,d\Gamma$ defines a variational formulation for the Neumann problem* (3.9).

Proof. Multiplying the PDE by $v \in V$ and using Green's formula, we easily see that if u solves problem (3.9), then we have for all $v \in V$, $a(u,v) = \ell(v)$.

Conversely, let us be given a function $u \in V$ such that for all $v \in V$, $a(u,v) = \ell(v)$. We would like to proceed as before and take $v \in \mathscr{D}(\Omega)$ to deduce the PDE. This does not work here because $\mathscr{D}(\Omega) \not\subset V$. For all $\varphi \in \mathscr{D}(\Omega)$, we set

$$\psi = \varphi - \frac{1}{\operatorname{meas}\Omega} \int_\Omega \varphi(x)\,dx,$$

[6] We have always said *the* variational formulation, but there is no evidence that it is unique in general.

so that $\psi \in V$ and we can use ψ as a test-function. Now φ and ψ differ by a constant $k = \frac{1}{\operatorname{meas}\Omega}\int_\Omega \varphi(x)\,dx$, therefore $\nabla\psi = \nabla\varphi$. We thus obtain,

$$\begin{aligned}\int_\Omega \nabla u\cdot\nabla\varphi\,dx &= \int_\Omega f\psi\,dx + \int_{\partial\Omega} g\psi\,d\Gamma\\ &= \int_\Omega f(\varphi+k)\,dx + \int_{\partial\Omega} g(\varphi+k)\,d\Gamma\\ &= \int_\Omega f\varphi\,dx + k\Big(\int_\Omega f\,dx + \int_{\partial\Omega} g\,d\Gamma\Big) = \int_\Omega f\varphi\,dx,\end{aligned}$$

since φ vanishes on $\partial\Omega$ and f,g satisfy condition (3.10). So we can deduce right away that $-\Delta u = f$ in the sense of distributions, and since $f \in L^2(\Omega)$ in the sense of $L^2(\Omega)$ as well.

We next pick an arbitrary $v \in V$ and apply Green's formula again. This yields

$$\int_\Omega fv\,dx + \int_{\partial\Omega} g\gamma_0(v)\,d\Gamma = -\int_\Omega(\Delta u)v\,dx + \int_{\partial\Omega}\gamma_1(u)\gamma_0(v)\,d\Gamma.$$

Hence, taking into account that $-\Delta u = f$, we obtain

$$\int_{\partial\Omega}(g-\gamma_1(u))\gamma_0(v)\,d\Gamma = 0$$

for all $v \in V$. Now it is clear that $\gamma_0(V) = H^{1/2}(\partial\Omega)$. Indeed, let is pick a $\theta \in \mathscr{D}(\Omega)$ such that $\int_\Omega \theta\,dx = 1$. Then, for all $w \in H^1(\Omega)$, $v = w - \int_\Omega w\,dx\theta \in V$ and $\gamma_0(v) = \gamma_0(w)$. Therefore, there are enough test-functions in V to conclude that $\gamma_1(u) = g$, since $H^{1/2}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$. □

Remark 3.3.3 We did not insist on the regularity needed to apply Green's formula or to define $\gamma_1(u)$ as an element of $H^{1/2}$, because it is possible to write down slightly more complicated arguments that completely do away with such artificial hypotheses. □

To apply the Lax-Milgram theorem, we need a new inequality.

Theorem 3.3.1 (Poincaré-Wirtinger inequality) *Let Ω be a Lipschitz open subset of $\mathbb{R}^d$. There exists a constant C depending on Ω such that, for all $v \in H^1(\Omega)$,*

$$\Big\|v - \frac{1}{\operatorname{meas}\Omega}\int_\Omega v\,dx\Big\|_{L^2(\Omega)} \le C\|\nabla v\|_{L^2(\Omega)}. \tag{3.11}$$

Proof. We admit the Poincaré-Wirtinger inequality. □

Remark 3.3.4 Even though there is a certain similarity with the Poincaré inequality, there are major differences. In particular, the Poincaré-Wirtinger inequality fails for open sets that are not regular enough (Lipschitz is sufficient for it to hold), whereas no regularity is needed for Poincaré. Instead of proving it, let us just note that the Poincaré-Wirtinger is at least reasonable, since both sides vanish for constant functions v. □

Proposition 3.3.5 *Let Ω be a Lipschitz open subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. Then the problem: Find $u \in V$ such that*

$$\forall v \in V, \quad \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g \gamma_0(v) \, d\Gamma,$$

has one and only one solution.

Proof. We have already shown that V is a Hilbert space for the H^1 scalar product. The continuity of both bilinear and linear forms have also already been proved. Only the V-ellipticity remains.

For all $v \in V$, we have $\int_\Omega v \, dx = 0$, hence by the Poincaré-Wirtinger inequality (3.11),

$$\|v\|^2_{H^1(\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} \leq (C^2+1)\|\nabla v\|^2_{L^2(\Omega)}.$$

Therefore,

$$a(v,v) = \|\nabla v\|^2_{L^2(\Omega)} \geq \alpha \|v\|^2_{H^1(\Omega)}$$

with $\alpha = \frac{1}{(C^2+1)} > 0$. □

Remark 3.3.5 The compatibility condition (3.10) plays no role in the application of the Lax-Milgram theorem. So exercise: What happens when it is not satisfied? What exactly are we solving then?

Since the space V is a hyperplane of H^1 that is L^2 orthogonal to the constants, it follows that the general solution of the Neumann problem is of the form $v+s$, where $v \in V$ is the unique solution of the variational problem above and $s \in \mathbb{R}$ is arbitrary. □

We now introduce a new kind of boundary condition, the Fourier condition (also called the Robin condition or the third boundary condition). The boundary value problem reads

$$\begin{cases} -\Delta u + cu = f \text{ in } \Omega, \\ \quad bu + \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega, \end{cases} \tag{3.12}$$

where b and c are given functions. When $b = 0$, we recognize the Neumann problem (and, in a sense, when $b = +\infty$ the Dirichlet problem). This condition is called after Fourier who introduced it in the context of the heat equation. In the heat interpretation, $\frac{\partial u}{\partial n}$ represents the heat flux through the boundary. Let us assume that we are modeling a situation in which the boundary is actually a very thin wall that insulates Ω from the outside where the temperature is $0°$. If $g = 0$, the Fourier condition states that $\frac{\partial u}{\partial n} = -bu$, that is to say that the heat flux passing through the wall is proportional to the temperature difference between the inside and the outside. For this interpretation to be physically reasonable, it is clearly necessary that $b \geq 0$, *i.e.*, the heat flows inwards when the outside is warmer than the inside and conversely. It thus to be expected that the sign of b will play a role.

We follow the same pattern as before: First find a variational formulation for the boundary value problem (3.12), second apply Lax-Milgram to prove existence and uniqueness.

Proposition 3.3.6 *Assume that $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, $c \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega)$. Then the triple*

$$\begin{aligned} V &= H^1(\Omega), \\ a(u,v) &= \int_\Omega (\nabla u \cdot \nabla v + cuv)\,dx + \int_{\partial\Omega} b\gamma_0(u)\gamma_0(v)\,d\Gamma, \\ \ell(v) &= \int_\Omega fv\,dx + \int_{\partial\Omega} g\gamma_0(v)\,d\Gamma, \end{aligned}$$

defines a variational formulation for the Fourier problem (3.12).

Proof. As always, we multiply the PDE by $v \in V$ and use Green's formula,

$$\begin{aligned} \int_\Omega (\nabla u \cdot \nabla v + cuv)\,dx &= \int_\Omega fv\,dx + \int_{\partial\Omega} \gamma_1(u)\gamma_0(v)\,d\Gamma \\ &= \int_\Omega fv\,dx + \int_{\partial\Omega} (g - b\gamma_0(u))\gamma_0(v)\,d\Gamma, \end{aligned}$$

hence

$$\int_\Omega (\nabla u \cdot \nabla v + cuv)\,dx + \int_{\partial\Omega} b\gamma_0(u)\gamma_0(v)\,d\Gamma = \int_\Omega fv\,dx + \int_{\partial\Omega} g\gamma_0(v)\,d\Gamma, \quad (3.13)$$

for all $v \in V$.

Conversely, let us be given a solution u of the variational problem (3.13). Taking first $v = \varphi \in \mathscr{D}(\Omega)$, all the boundary integrals vanish and we obtain $-\Delta u + cu = f$ exactly as before. Taking then v arbitrary, using Green's formula and the PDE just obtained, we get

$$\int_{\partial\Omega} \gamma_1(u)\gamma_0(v)\,d\Gamma + \int_{\partial\Omega} b\gamma_0(u)\gamma_0(v)\,d\Gamma = \int_{\partial\Omega} g\gamma_0(v)\,d\Gamma,$$

so that

$$\int_{\partial\Omega} (\gamma_1(u) + b\gamma_0(u) - g)\gamma_0(v)\,d\Gamma = 0,$$

for all $v \in V = H^1(\Omega)$, hence the Fourier boundary condition. □

Remark 3.3.6 A natural question to ask is why not keep the term $\gamma_1(u)$ in the bilinear form? The answer is that, while it is true that $\gamma_1(u)$ exists when u is a solution of either the boundary value problem or the variational problem, it does not exist for a general $v \in H^1(\Omega)$, hence cannot appear in a bilinear form that is defined on $H^1(\Omega) \times H^1(\Omega)$. Besides, how would b appear otherwise? □

Let us give a first existence and uniqueness result.

Proposition 3.3.7 *Let Ω be a Lipschitz open subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, $c \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega)$. Assume that $c \geq \eta > 0$ for some constant η and that $\|b_-\|_{L^\infty(\partial\Omega)} < \frac{\min(1,\eta)}{C_{\gamma_0}^2}$, where C_{γ_0} is the continuity constant of the trace mapping. Then the problem: Find $u \in V$ such that*

$$\forall v \in V, \int_\Omega (\nabla u \cdot \nabla v + cuv)\,dx + \int_{\partial\Omega} b\gamma_0(u)\gamma_0(v)\,d\Gamma = \int_\Omega fv\,dx + \int_{\partial\Omega} g\gamma_0(v)\,d\Gamma,$$

has one and only one solution.

Here $b_- = -\min(0,b)$ denotes the negative part of b.

Proof. We check the hypotheses of the Lax-Milgram theorem. We already know that V is a Hilbert space. The continuity of the bilinear form a has also already been checked, except for the boundary integral terms

$$\begin{aligned}\left|\int_{\partial\Omega} b\gamma_0(u)\gamma_0(v)\,d\Gamma\right| &\leq \|b\|_{L^\infty(\partial\Omega)}\|\gamma_0(u)\|_{L^2(\partial\Omega)}\|\gamma_0(v)\|_{L^2(\partial\Omega)}\\ &\leq C_{\gamma_0}^2\|b\|_{L^\infty(\partial\Omega)}\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}\end{aligned}$$

for all u and v. The linear form is also known to be continuous. Let us check the V-ellipticity. Obviously $b \geq -b_-$, thus

$$\begin{aligned}&\int_\Omega (\|\nabla v\|^2 + cv^2)\,dx + \int_{\partial\Omega} b\gamma_0(v)^2\,d\Gamma\\ &\qquad\geq \min(1,\eta)\|v\|_{H^1(\Omega)}^2 - \|b_-\|_{L^\infty(\partial\Omega)}\|\gamma_0(v)\|_{L^2(\partial\Omega)}^2\\ &\qquad\qquad\geq \big(\min(1,\eta) - C_{\gamma_0}^2\|b_-\|_{L^\infty(\partial\Omega)}\big)\|v\|_{H^1(\Omega)}^2,\end{aligned}$$

hence the V-ellipticity. □

Remark 3.3.7 Under the previous hypotheses, we have existence and uniqueness via Lax-Milgram provided b is not too negative in some sense. □

All these hypotheses only give sufficient conditions. Let us give another set of such hypotheses.

Proposition 3.3.8 *Same hypotheses except that we assume that $c \geq 0$ and that $b \geq \mu > 0$ for some constant μ. Then the Fourier problem has one and only one solution.*

Proof. The only point to be established is V-ellipticity. We use a compactness argument by contradiction. For this we admit that Rellich's theorem 2.7.3 is also true in dimension d in a Lipschitz open set. We have

$$\int_\Omega (\|\nabla v\|^2 + cv^2)\,dx + \int_{\partial\Omega} b\gamma_0(v)^2\,d\Gamma \geq \int_\Omega \|\nabla v\|^2\,dx + \mu\int_{\partial\Omega}\gamma_0(v)^2\,d\Gamma.$$

Let us assume for contradiction that there is no constant $\alpha > 0$ such that

$$\int_\Omega \|\nabla v\|^2\,dx + \mu\int_{\partial\Omega}\gamma_0(v)^2\,d\Gamma \geq \alpha\|v\|^2_{H^1(\Omega)}.$$

This implies that for all $n \in \mathbb{N}^*$, there exists $v_n \in H^1(\Omega)$ such that

$$\int_\Omega \|\nabla v_n\|^2\,dx + \mu\int_{\partial\Omega}\gamma_0(v_n)^2\,d\Gamma < \frac{1}{n}\|v_n\|^2_{H^1(\Omega)}.$$

We can assume without loss of generality that

$$\|v_n\|^2_{H^1(\Omega)} = 1, \tag{3.14}$$

and we have

$$\int_\Omega \|\nabla v_n\|^2\,dx + \mu\int_{\partial\Omega}\gamma_0(v_n)^2\,d\Gamma \to 0. \tag{3.15}$$

Now v_n is bounded in $H^1(\Omega)$ by (3.14), thus relatively compact in $L^2(\Omega)$ by Rellich's theorem. We may extract a subsequence, still denoted v_n, and $v \in L^2(\Omega)$ such that $v_n \to v$ in $L^2(\Omega)$. By (3.15), $\|\nabla v_n\|_{L^2(\Omega)} \to 0$, therefore, since $\nabla v_n \to \nabla v$ in $\mathscr{D}'(\Omega)$, we have $\nabla v = 0$ and v is constant on each connected component of Ω. Moreover,

$$\|v_n - v\|^2_{H^1(\Omega)} = \|\nabla v_n\|^2_{L^2(\Omega)} + \|v_n - v\|^2_{L^2(\Omega)} \to 0 \tag{3.16}$$

so that, by continuity of the trace mapping $\gamma_0(v_n) \to \gamma_0(v)$ in $L^2(\partial\Omega)$. By (3.15) again, we also have $\|\gamma_0(v_n)\|_{L^2(\partial\Omega)} \to 0$ since $\mu > 0$ and therefore $\gamma_0(v) = 0$. It follows that v being a constant with zero trace vanishes in each connected component, *i.e.*, $v = 0$. We now realize that (3.14) and (3.16) contradict each other, therefore our premise that there exists no V-ellipticity constant α is false. □

Remark 3.3.8 This is a typical compactness argument: we can prove that the constant exists but we have no idea of its value. □

3.4 General second order elliptic problems

Up to now, the partial differential operator always was the Laplacian. Let us rapidly consider more general second order elliptic operators in a Lipschitz open subset Ω of $\mathbb{R}^d$. We are given a $d \times d$ matrix-valued function $A(x) = (a_{ij}(x))$ with $a_{ij} \in C^1(\bar{\Omega})$. Let $u \in C^2(\Omega)$ (we can lower this regularity considerably), then $A\nabla u$ is a vector field with components

$$(A\nabla u)_i = \sum_{j=1}^{d} a_{ij}\partial_j u$$

whose divergence is given by

$$\begin{aligned}\operatorname{div}(A\nabla u) &= \sum_{i=1}^{d} \partial_i (A\nabla u)_i \\ &= \sum_{i,j=1}^{d} a_{ij}\partial_{ij}u + \sum_{j=1}^{d}\Big(\sum_{i=1}^{d} \partial_i a_{ij}\Big)\partial_j u.\end{aligned}$$

The principal part of this operator $\sum_{i,j=1}^{d} a_{ij}\partial_{ij}$ is of the second order. We will consider the boundary value problem

$$\begin{cases} -\operatorname{div}(A\nabla u) + cu = f \text{ in } \Omega, \\ u = h \text{ on } \Gamma_0, \\ bu + n\cdot A\nabla u = g \text{ on } \Gamma_1, \end{cases} \tag{3.17}$$

where c, b, f, g and h are given functions and Γ_0, Γ_1 a partition of $\partial\Omega$ as in the mixed problem. When $A = I$, we recognize $-\operatorname{div}(A\nabla u) = -\Delta u$ and $n\cdot A\nabla u = \frac{\partial u}{\partial n}$, so that we are generalizing all the model problems seen up to now. First of all, we reduce the study to the case $h = 0$ by subtracting a function with the appropriate trace, as before.

Proposition 3.4.1 *Assume that $f \in L^2(\Omega)$, $g \in L^2(\Gamma_1)$, $c \in L^\infty(\Omega)$ and $b \in L^\infty(\Gamma_1)$. Then the triple*

$$\begin{aligned} V &= \{v \in H^1(\Omega); \gamma_0(v) = 0 \textit{ on } \Gamma_1\}, \\ a(u,v) &= \int_\Omega (A\nabla u\cdot\nabla v + cuv)\,dx + \int_{\Gamma_1} b\gamma_0(u)\gamma_0(v)\,d\Gamma, \\ \ell(v) &= \int_\Omega fv\,dx + \int_{\Gamma_1} g\gamma_0(v)\,d\Gamma, \end{aligned}$$

defines a variational formulation for problem (3.17).

Proof. The proof is routine, but we write it partially down for completeness. Multiply the PDE by $v \in V$ and integrate by parts. This yields first

$$-\int_\Omega \Big(\sum_{i=1}^d \partial_i (A\nabla u)_i\Big) v\,dx + \int_\Omega cuv\,dx = \int_\Omega fv\,dx,$$

then

$$\int_\Omega \sum_{i=1}^d (A\nabla u)_i \partial_i v\,dx - \int_{\Gamma_1} \Big(\sum_{i=1}^d (A\nabla u)_i n_i\Big) \gamma_0(v)\,d\Gamma + \int_\Omega cuv\,dx = \int_\Omega fv\,dx,$$

and finally

$$\int_\Omega (A\nabla u\cdot\nabla v + cuv)\,dx + \int_{\Gamma_1} b\gamma_0(u)\gamma_0(v)\,d\Gamma = \int_\Omega fv\,dx + \int_{\Gamma_1} g\gamma_0(v)\,d\Gamma.$$

We leave the converse to the reader. □

Proposition 3.4.2 *Let Ω be a Lipschitz open subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma_1)$, $c \in L^\infty(\Omega)$ and $b \in L^\infty(\Gamma_1)$. We assume that the matrix A is* uniformly elliptic*, that is to say that there exists a constant $\alpha > 0$ such that*

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \alpha\|\xi\|^2$$

for all $x \in \bar\Omega$ and all $\xi \in \mathbb{R}^d$. We assume in addition that $c \geq \eta > 0$ for some constant η and that $b \geq 0$. Then the problem: Find $u \in V = \{v \in H^1(\Omega); \gamma_0(v) = 0$ on $\Gamma_1\}$ such that

$$\forall v \in V, \int_\Omega (A\nabla u\cdot\nabla v + cuv)\,dx + \int_{\Gamma_1} b\gamma_0(u)\gamma_0(v)\,d\Gamma = \int_\Omega fv\,dx + \int_{\Gamma_1} g\gamma_0(v)\,d\Gamma,$$

has one and only one solution.

Proof. That V is a Hilbert space and that ℓ is continuous are already known facts. We leave the proof of the continuity of the bilinear form, which is implied by the boundedness of the matrix coefficients $a_{ij}(x)$. The V-ellipticity is also quite obvious, since

$$\begin{aligned} a(v,v) &= \int_\Omega (A\nabla v\cdot\nabla v + cv^2)\,dx + \int_{\Gamma_1} b\gamma_0(v)^2\,d\Gamma \\ &\geq \alpha\int_\Omega \|\nabla v\|^2\,dx + \eta\int_\Omega v^2\,dx \\ &\geq \min(\alpha,\eta)\|v\|^2_{H^1(\Omega)}, \end{aligned}$$

hence the existence, uniqueness and continuous dependence of the solution on the data by Lax-Milgram. □

Remark 3.4.1 When the matrix A is not symmetric, neither is the bilinear form a, even though the principal part of the operator is symmetric since $\sum_{i,j=1}^d a_{ij}\partial_{ij} = \sum_{i,j=1}^d \frac{a_{ij}+a_{ji}}{2}\partial_{ij}$ due to the fact that $\partial_{ij} = \partial_{ji}$. When A is symmetric, then so is the bilinear form and we have an equivalent minimization problem with

$$J(v) = \frac{1}{2}\Big[\int_\Omega (A\nabla v\cdot\nabla v + cv^2)\,dx + \int_{\Gamma_1} b\gamma_0(v)^2\,d\Gamma\Big] - \int_\Omega fv\,dx - \int_{\Gamma_1} g\gamma_0(v)\,d\Gamma,$$

minimized over V.

It is quite clear that we can reduce the regularity of A down to L^∞ without loosing the existence and uniqueness of the variational problem. The interpretation in terms of PDEs stops at the divergence form $-\mathrm{div}\,(A\nabla u) + cu = f$ since we cannot develop the divergence using Leibniz formula in this case. Such lack of regularity is useful to model heterogeneous media. □

We now give an example of a non symmetric problem, the convection-diffusion problem. Let us be given a vector field σ. The convection-diffusion problem reads

$$\begin{cases} -\Delta u + \sigma\cdot\nabla u + cu = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.18}$$

We have a diffusion term $-\Delta u$ and a transport term $\sigma\cdot\nabla u$ in the same equation that compete with each other.

Proposition 3.4.3 *Assume that $f \in L^2(\Omega)$, $\sigma \in C^1(\bar\Omega;\mathbb{R}^d)$ and $c \in L^\infty(\Omega)$. Then the triple*

$$\begin{aligned} V &= H_0^1(\Omega), \\ a(u,v) &= \int_\Omega \big(\nabla u\cdot\nabla v + (\sigma\cdot\nabla u + cu)v\big)\,dx, \\ \ell(v) &= \int_\Omega fv\,dx, \end{aligned}$$

defines a variational formulation for problem (3.18).

Proof. This is really routine now... Note that the bilinear form is not symmetric.□

Proposition 3.4.4 *Let Ω be a bounded open subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$, $\sigma \in C^1(\bar\Omega;\mathbb{R}^d)$ and $c \in L^\infty(\Omega)$. We assume that $c - \frac{1}{2}\mathrm{div}\,\sigma \geq 0$. Then the problem: Find $u \in V$ such that*

$$\forall v \in V, \int_\Omega \big(\nabla u\cdot\nabla v + (\sigma\cdot\nabla u + cu)v\big)\,dx = \int_\Omega fv\,dx,$$

has one and only one solution.

Proof. We just prove the V-ellipticity. We have

$$a(v,v) = \int_\Omega \big(\|\nabla v\|^2 + cv^2 + (\sigma\cdot\nabla v)v\big)\,dx.$$

Let us integrate the last integral by parts

$$\begin{aligned}\int_\Omega (\sigma\cdot\nabla v)v\,dx &= \int_\Omega \Big(\sum_{i=1}^d \sigma_i\partial_i v\Big)v\,dx\\ &= -\int_\Omega \Big(\sum_{i=1}^d \partial_i(\sigma_i v)\Big)v\,dx = -\int_\Omega \Big(\sum_{i=1}^d \partial_i\sigma_i\Big)v^2\,dx - \int_\Omega \Big(\sum_{i=1}^d \sigma_i\partial_i v\Big)v\,dx\\ &= -\int_\Omega \operatorname{div}\sigma v^2\,dx - \int_\Omega (\sigma\cdot\nabla v)v\,dx,\end{aligned}$$

since all boundary terms vanish, so that

$$\int_\Omega (\sigma\cdot\nabla v)v\,dx = -\frac{1}{2}\int_\Omega \operatorname{div}\sigma v^2\,dx.$$

Therefore

$$a(v,v) = \int_\Omega \Big(\|\nabla v\|^2 + \Big(c - \frac{1}{2}\operatorname{div}\sigma\Big)v^2\Big)\,dx \geq |v|^2_{H^1(\Omega)},$$

hence the result by the equivalence of the H^1 semi-norm and the H^1 norm on $H^1_0(\Omega)$. □

Remark 3.4.2 We thus have existence and uniqueness if $c = 0$ and $\operatorname{div}\sigma = 0$. The case $\operatorname{div}\sigma = 0$ is interesting because if σ represents the velocity field of such a fluid as air or water, the divergence free condition is the expression of the incompressibility of the fluid. Under usual experimental conditions, both fluids are in fact considered to be incompressible. □

Let us now give a fourth order example, even though only second order problems were advertised in the section title. We consider a slight variant of the plate problem with homogeneous Dirichlet boundary conditions

$$\begin{cases}\Delta^2 u + cu = f \text{ in } \Omega,\\ u = 0 \text{ on } \partial\Omega,\\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,\end{cases} \tag{3.19}$$

with Ω bounded in $\mathbb{R}^d$ ($d = 2$ in the actual case of a plate).

The derivation of a variational formulation is again fairly routine, but since this is our first (and only) fourth order problem, we give some detail. The variational space for this Dirichlet problem is $V = H_0^2(\Omega)$ which incorporates the two boundary conditions. Assume that $u \in H^4(\Omega) \cap H_0^2(\Omega)$. Then $\Delta u \in H^2(\Omega)$ and we can use Green's formula

$$\begin{aligned}\int_\Omega (\Delta^2 u) v\, dx &= \int_\Omega (\Delta(\Delta u)) v\, dx \\ &= \int_\Omega \Delta u \Delta v\, dx + \int_{\partial\Omega} (\gamma_0(v)\gamma_1(\Delta u) - \gamma_1(v)\gamma_0(\Delta u))\, d\Gamma \\ &= \int_\Omega \Delta u \Delta v\, dx\end{aligned}$$

since $\gamma_0(v) = \gamma_1(v) = 0$ for all $v \in H_0^2(\Omega)$. So we have our variational formulation

$$\forall v \in V, \quad \int_\Omega (\Delta u \Delta v + cuv)\, dx = \int_\Omega f v\, dx, \tag{3.20}$$

which is easily checked to give rise to a solution of the boundary value problem.

Let $\nabla^2 v$ denote the collection of all d^2 second order partial derivatives of v. We have

Lemma 3.4.1 *The semi-norm $\|\nabla^2 v\|_{L^2(\Omega)}$ is a norm on $H_0^2(\Omega)$ that is equivalent to the H^2 norm.*

Proof. It is enough to establish a bound from below. Let $v \in H_0^2(\Omega)$. Then we have $\partial_i v \in H_0^1(\Omega)$ for all i. Therefore $\|\nabla(\partial_i v)\|_{L^2(\Omega)}^2 \geq C^2 \|\partial_i v\|_{H^1(\Omega)}^2$, by Poincaré. Now of course

$$\|\partial_i v\|_{H^1(\Omega)}^2 = \|\nabla(\partial_i v)\|_{L^2(\Omega)}^2 + \|\partial_i v\|_{L^2(\Omega)}^2,$$

so summing in i, we get

$$\begin{aligned}\|\nabla^2 v\|_{L^2(\Omega)}^2 = \sum_{i=1}^d \|\nabla(\partial_i v)\|_{L^2(\Omega)}^2 &\geq C^2 \Big(\|\nabla^2 v\|_{L^2(\Omega)}^2 + |v|_{H^1(\Omega)}^2 \Big) \\ &\geq C^2 \|\nabla^2 v\|_{L^2(\Omega)}^2 + C^4 \|v\|_{H^1(\Omega)}^2 \geq C^4 \|v\|_{H^2(\Omega)}^2\end{aligned}$$

since $v \in H_0^1(\Omega)$ and $C \leq 1$. □

Proposition 3.4.5 *Let Ω be a bounded open subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$ and $c \in L^\infty(\Omega)$. We assume that $c \geq 0$. Then problem* (3.20) *has one and only one solution.*

Proof. We just prove the V-ellipticity. We have

$$a(v,v) \geq \int_\Omega (\Delta v)^2\,dx.$$

We argue by density. Let $\varphi \in \mathscr{D}(\Omega)$, since $\Delta\varphi = \sum_{i=1}^d \partial_{ii}\varphi$, we can write

$$\begin{aligned}\int_\Omega (\Delta\varphi)^2\,dx &= \int_\Omega \Big(\sum_{i=1}^d \partial_{ii}\varphi\Big)\Big(\sum_{j=1}^d \partial_{jj}\varphi\Big)\,dx = \sum_{i,j=1}^d \int_\Omega \partial_{ii}\varphi\partial_{jj}\varphi\,dx \\ &= -\sum_{i,j=1}^d \int_\Omega \partial_i\varphi\partial_{ijj}\varphi\,dx = \sum_{i,j=1}^d \int_\Omega \partial_{ij}\varphi\partial_{ij}\varphi\,dx\end{aligned}$$

with two successive integrations by parts, the first with respect to x_i and the second with respect to x_j. Hence, for all $\varphi \in \mathscr{D}(\Omega)$, we obtain

$$\int_\Omega (\Delta\varphi)^2\,dx = \sum_{i,j=1}^d \int_\Omega (\partial_{ij}\varphi)^2\,dx = \|\nabla^2\varphi\|^2_{L^2(\Omega)}.$$

Now, by definition, $H_0^2(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ in $H^2(\Omega)$, thus for all $v \in H_0^2(\Omega)$, there exists a sequence $\varphi_n \in \mathscr{D}(\Omega)$ such that $\varphi_n \to v$ in $H^2(\Omega)$. Passing to the limit in the above equality, we thus get

$$\int_\Omega (\Delta v)^2\,dx = \|\nabla^2 v\|^2_{L^2(\Omega)},$$

since $\partial_{ij}\varphi_n \to \partial_{ij}v$ in $L^2(\Omega)$, hence the result by Lemma 3.4.1. □

Remark 3.4.3 Notice the trick used in the above proof. To establish an equality for H^2 functions, we need to use third derivatives, which make no sense as functions in this context. However, the formula is valid for smooth functions, for which third derivatives are not a problem, and since in the end the formula in question does not involve any derivatives of order higher than two, it extends to H^2 by density.

The formula is actually surprising, since Δv does not contain any derivative $\partial_{ij}v$ with $i \neq j$, and only the sum of all $\partial_{ii}v$ derivatives. Its L^2 norm squared is nonetheless equal to the sum of the L^2 norms squared of all individual second derivatives. This is related to elliptic regularity, which was mentioned in passing before. □

Remark 3.4.4 This is another symmetric problem, hence we have an equivalent energy minimization formulation with

$$J(v) = \frac{1}{2}\int_\Omega (\Delta v)^2\,dx - \int_\Omega fv\,dx,$$

to be minimized on $H_0^2(\Omega)$. □

To conclude this section, we discuss the general three point strategy for solving elliptic problems that was repeatedly applied here. First we establish a variational formulation: (homogeneous) Dirichlet boundary conditions are enforced by the test-function space, which is included in H^1 for second order problems; we multiply the PDE by a test-function—possibly assuming additional regularity on the solution—and use integration by parts or Green's formula to obtain the variational problem. The bilinear form must be well-defined on the test-function space.

The second point is to check that the variational formulation actually gives rise to a solution of the boundary value problem. This point is usually itself in two steps: first obtain the PDE in the sense of distributions by using test-functions in $\mathscr{D}$, second retrieve Neumann or Fourier boundary conditions by using the full test-function space. The first two points can appear somewhat formal because of the assumed regularity on the solution that is not always easily obtained in the end. This is not a real problem, since it is possible to write rigorous arguments, at the expense of more theory than we need here.

The final third point is to try and apply the Lax-Milgram theorem, by making precise regularity and possibly sign assumptions on the various functions that act as data. Here we prove existence and uniqueness of the solution to the variational problem.

A question that can be asked is what is the relevance of such solutions to a boundary value problem, in which the partial derivatives are taken in a rather weak sense. This is where elliptic regularity theory comes into play. Using elliptic regularity, it is possible to show that the variational solution is indeed the classical solution, provided the data (coefficients, right-hand side, boundary of Ω) is smooth enough. □