# Contents

# 1 Real numbers

## 1.1 Preliminaries

**Definition 1.1.** 1. A set is a well-defined collection of distinct objects, called the elements or members of the set. Sets may be finite or infinite. They are typically denoted by curly braces { } and listing the elements separated by commas.

2. The empty set denoted by $\phi$ is a set that has no elements.

3. If $x$ is an element of the set $A$, we write $x \in A$, if not we write $x \notin A$.

4. A set $A$ is subset of $B$ or $A$ is included in $B$ if every element of $A$ belongs to $B$ and we write $A \subset B$, that is,

$$\boxed{x \in A \Longrightarrow x \in B.}$$

5. Tow sets $A$ and $B$ are equals if its have the same elements and we write $A = B$. In other terms $A = B$ if $A \subset B$ and $B \subset A$, or

$$\boxed{x \in A \Longleftrightarrow x \in B.}$$

☞ **Example 1.1.**
- $A = \{1, 2, 3\}$ is a set containing the members 1, 2, and 3 (finite set).
- $A = \{0, 2, 4, 6, ...\}$ is a set of positive even integers (infinite set).
- $A = \{\frac{n^2+1}{n+1} \mid n \in \mathbb{N}\}$ is a set where the element are given by the expression $\frac{n^2+1}{n+1}$ for all $n \in \mathbb{N}$. We have $0 \notin A$, $1 \in A$ because $1 = \frac{1^2+1}{1+1}$, $2 \notin A$ because $2 \neq \frac{n^2+1}{n+1}$ for all $n \in \mathbb{N}$.
- $A = \{x \in \mathbb{R} : x^2 + 3x + 1 \leq 0\}$ is a set containing the solutions of the inequality $x^2 + x + 1 \leq 0$. For example, $0 \notin A$ because $0^2 + 3 \times 0 + 1 = 1 \nleq 0$, $-1/2 \in A$ because $(-1/2)^2 + 3(-1/2) + 1 = -1/4 \leq 0$.

**Definition 1.2.** 1. The set of natural numbers denoted by $\mathbb{N}$ is defined by

$$\boxed{\mathbb{N} := \{0, 1, 2, 3, \cdots\}}$$

2. The set of integers denoted by $\mathbb{Z}$ is defined by

$$\boxed{\mathbb{Z} := \{\cdots - 2, -1, 0, 1, 2, \cdots\}}$$

3. Endowed by the operation of addition " + ", the set of integers is an Abelien group. That is is

   - Closure: For all $x, y \in E$, $x + y \in \mathbb{Z}$.

- $+$ is commutative : $\forall x, y \in \mathbb{Z} : x + y = y + x$
- $+$ is associative : $\forall x, y, z \in \mathbb{Z} : (x + y) + z = x + (y + z)$
- **Identity Element**: There exists an element $0 \in \mathbb{Z}$ such that $\forall x \in \mathbb{Z} : x + 0 = x$.
- **symmetric Element**: For every $x \in E$, there exists an element $-x \in \mathbb{Z}$ such that $x + (-x) := x - y = 0$.

**Definition 1.3 (Ordered sets).** An ordered set is a set $E$ endowed by a relation " $<$ " such that

- For all $x, y \in E$, exactly one of the following holds

$$x < y, \ x = y, \ \text{or } y < x$$

- For all $x, y, z \in E : x < y \wedge y < z \Longrightarrow x < z$ (transitivity)

We write $x \leq y$ if $x < y$ or $x = y$.

☞ **Example 1.2.** • The set of natural numbers $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$ and the set of integers $\mathbb{Z} := \{\cdots - 2, -1, 0, 1, 2, \cdots\}$ are ordered sets with the relation (lower than) $<$ and we have

$$\cdots - 3 < -2 < -1 < 0 < 1 < 2 < 3 < \cdots$$

**Definition 1.4.** Let $(E, <)$ be an ordered set and let $A$ be a subset of $E$.

- We say $a \in E$ is an lower-bound of $A$ if

$$\forall x \in A : a \leq x$$

  and if there exist an lower-bound of $A$, we say $A$ is bounded below.

- We say $b \in E$ is an upper-bound of $A$ if

$$\forall x \in A : x \leq b$$

  and if there exist an upper-bound of $A$, we say $A$ is bounded above.

- We say $a_0 \in E$ is the greatest lower-bound or the infimum of $A$ if $a_0$ is an lower-bound of $A$ and satisfies $a \leq a_0$ for every lower-bound $a \in E$. We write

$$a_0 := \inf A$$

- We say $b_0 \in E$ is the least upper-bound or the supremum of $A$ if $b_0$ is an upper-bound of $A$ and satisfies $b_0 \leq b$ for every upper-bound $b \in E$. We write

$$b_0 := \sup A$$

☞ **Example 1.3.**

**Definition 1.5.** The set of rational numbers is the set denoted by $\mathbb{Q}$ defined as follows

$$\mathbb{Q} = \{\tfrac{p}{q} \mid (p,q) \in \mathbb{Z} \times \mathbb{Z}^*\}$$

or

$$\mathbb{Q} = \{\tfrac{p}{q} \mid (p,q) \in \mathbb{Z} \times \mathbb{N}^*\}$$

or

$$\mathbb{Q} = \{\tfrac{p}{q} \mid (p,q) \in \mathbb{Z} \times \mathbb{N}^*, \text{ with } p \wedge q = 1\}$$

☞ **Remark 1.1.** The set of rational numbers $\mathbb{Q}$ is an ordered set with the relation $<$ "lower than" defined as follow

$$x < y \Longleftrightarrow y - x = p/q \text{ where } p, q \in \mathbb{N}$$

and then we say that $y - x$ is negative, if it is not positive, we say that it is negative.

**Definition 1.6.** The addition and multiplicative operations on $\mathbb{Q}$ are defined as follow

$$\tfrac{p}{q} + \tfrac{p'}{q'} = \tfrac{pq'+qp'}{p'q'}, \quad \tfrac{p}{q} \cdot \tfrac{p'}{q'} = \tfrac{pp'}{qq'}, \quad \text{for all } p \in \mathbb{Z}, q \in \mathbb{Z}^*.$$

**Theorem 1.1.** The set of rational numbers $\mathbb{Q}$ endowed with the addition and multiplicative operations is an abilean field. That is

1. $(\mathbb{Q}, +)$ **is an abeliean group**
2. **Multiplicative Associativity**: For all $x, y, z \in \mathbb{Q}$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
3. **Multiplicative Identity Element**: There exists an element $1 \in \mathbb{Q}$ such that for all $x \in \mathbb{Q}$, $x \cdot 1 = 1 \cdot x = x$.
4. **Multiplicative Inverse Element (except for 0)**: For every non-zero $x \in \mathbb{Q}$, there exists an element $x^{-1} \in \mathbb{Q}$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$.
5. **Distributive Property**: For all $x, y, z \in \mathbb{Q}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

**Definition 1.7 (least upper bound property).** Le $E$ be an ordered set.

1. We say that $E$ satisfies the least upper bound property if every non empty subset $A$ of $E$ that is bounded from above has the least upper bound (i.e. $\sup A$ exists in $E$).
2. We say that $E$ satisfies the greatest lower bound property if every non empty subset $A$ of $E$ that is bounded from below has the greatest lower bound (i.e. $\inf A$ exists in $E$)

☞ **Remark 1.2.** The ordered set $\mathbb{Q}$ does not satisfy the least upper bound property. Indeed consider the following subset of $\mathbb{Q}$ :

$$A = \{x \in \mathbb{Q} : x^2 \leq 2\}.$$

This set is bounded above by 2 because for every $x \in A$ we have $x \leq 2$ (if not then $x^2 \geq 4$ and $x \notin A$). Suppose by absurd that $A$ has a least upper bound denoted by $b$. Assume, for the sake of contradiction, that the set $A = \{x \in \mathbb{Q} : x^2 \leq 2\}$ has a least upper bound $\alpha$ in $\mathbb{Q}$. We divide the proof in two steps

- We claim that $\alpha^2 = 2$. Indeed, if $\alpha^2 > 2$, then for $h := \frac{\alpha^2-2}{2\alpha}$, we have $\alpha - h < \alpha$ and

  $$(\alpha - h)^2 = \alpha^2 - 2\alpha h + h^2 > \alpha^2 - 2\alpha h = 2.$$

  Thus, $\alpha - h$ is an upper bound of $A$, which contradicts the fact that $\alpha = \sup A$. If $\alpha^2 < 2$, then for $h := \min\{1, \frac{2-\alpha^2}{2\alpha+1}\} \in \mathbb{Q}$, we have $\alpha < \alpha + h$ and

  $$(\alpha + h)^2 = \alpha^2 + 2\alpha h + h^2 \leq \alpha^2 + 2\alpha h + h \leq 2.$$

  Thus $\alpha + h \in A$ and $\alpha < \alpha + h$. Then $\alpha$ is not an upper bound. Contradiction. Hence $\alpha^2 = 2.$

- Let us show that $\alpha \notin \mathbb{Q}$. If not then $\alpha = \frac{p}{q}$, where $p$ and $q$ are integers with no common factors other than 1. Hence $2 = \alpha^2 = \frac{p^2}{q^2}$ and $p^2 = 2q^2$. This implies that $p^2$ is an even number, and therefore, $p$ is also be even (because the square of an odd number is odd). So we can write $p$ as $p = 2k$ where $k$ is an integer. Therefore $2q^2 = (2k)^2$. It follows that $q^2 = 2k^2$ is even and also is $q$ However, this contradicts our initial assumption that $p$ and $q$ have no common factors other than 1, as both $p$ and $q$ are even. Consequently $\alpha$ is not rational number

☞ **Remark 1.3.** The ordered set $\mathbb{Z}$ has the least upper bound property and for every bounded set $A$ of $\mathbb{Z}$, we have

$$\sup A \in A, \ \inf A \in A$$

## 1.2 The set of real numbers

We have seen in the previous remark that the set of rational numbers $\mathbb{Q}$ haven't the least upper bound property. So we need an other set larger than $\mathbb{Q}$, that satisfies this property. This set is the real number set $\mathbb{R}$ given by the following definition

**Definition 1.8.** The real number set $\mathbb{R}$ is an ordered field containing $\mathbb{Q}$ and satisfies the least upper bound property.

The following theorem guaranties the existence of $\mathbb{R}$.

**Theorem 1.2.** There is a unique ordered field which extends the field of rational numbers $\mathbb{Q}$ and satisfies the least upper bound property.

*Proof.* is accepted. □

## 1.3 Absolute value

**Definition 1.9.** The absolute value denoted by $|\cdot|$ is a function defined from $\mathbb{R}$ to $\mathbb{R}_+$ as follows

$$\forall x \in \mathbb{R} : |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

or

$$\forall x \in \mathbb{R} : |x| = \max\{x, -x\}$$

**Proposition 1.3.** for all $x, y \in \mathbb{R}$, we have

1. $|x| = |-x|$, $|xy| = |x||y|$
2. $|x| \leq y \Longleftrightarrow -|y| \leq x \leq |y|$, $|x| \geq y \geq 0 \Longleftrightarrow x \leq -y \vee x \geq y$
3. $-|x| \leq x \leq |x|$
4. $\big|x + y\big| \leq |x| + |y|$ (**Triangle inequality**)
5. $\big||x| - |y|\big| \leq |x - y|$

## 1.4 Archimedean property, density and integer part property

**Definition 1.10.** Let $x \in \mathbb{R}$

1. The integer part of $x$ denoted as $[x]$ is the unique integer satisfying

$$[x] \leq x < [x] + 1$$

   or equivalently

$$x - 1 < [x] \leq x.$$

2. A set $A$ is said to be dense in $\mathbb{R}$ if

$$\forall x, y \in \mathbb{R}, x < y, \ \exists z \in A : x < z < y.$$

☞ **Example 1.4.**

- $[0.5] = 0$ because $0 \leq 0.5 < 1$.
- $[-1.5] = -2$ because $-2 \leq -1.5 < -1$.
- If $x \in \mathbb{Z}$ then $[x] = x$ because $x \leq x < x + 1$.

**Theorem 1.4 (Archimedean property).** we have

$$\forall x \in \mathbb{R}_+^*, y \in \mathbb{R}, \exists n \in \mathbb{N} : nx \geq y.$$

*Proof.* Divide through by $x$. Then the Archimedean property says that for every real number $a = \frac{y}{x}$ , we can find $n \in \mathbb{N}$ such that $n \geq a$. In other words, says that the set of natural numbers $\mathbb{N}$ is not bounded above. Suppose for contradiction that $\mathbb{N}$ is bounded above. Then due to the least upper bound axiom, there is $b = \sup \mathbb{N}$. Therefore number $b - 1$ cannot be an upper bound for $\mathbb{N}$ as it is strictly less than $b$ (the least upper bound). Thus there exists an $m \in \mathbb{N}$ such that $m > b - 1$. it follows that $n := m + 1 > b$. This is contradiction since $b$ being an upper bound. □

**Theorem 1.5.** The following properties are equivalent

1. **Archimedean property** $\forall x \in \mathbb{R}_+^*, y \in \mathbb{R}, \exists n \in \mathbb{N} : nx \geq y$.
2. **integer part property:** $\forall x \in \mathbb{R}, \exists n \in \mathbb{Z} : n \leq x < n + 1$
3. $\mathbb{Q}$ **is dense in** $\mathbb{R}$, that is $\forall x, y \in \mathbb{R}, x < y, \exists r \in \mathbb{Q} : x < r < y.$

*Proof.*

- 1) $\Longrightarrow$ 2) Let $x \in \mathbb{R}$ be given. We want to show that there exists an integer $n \in \mathbb{Z}$ such that $n \leq x < n + 1$. Consider the set

$$S = \{n \in \mathbb{Z} : n \leq x\}.$$

Due to the Archimedean property, the set $S$ is non empty. Indeed. There is $n \in \mathbb{Z} : -n \geq -x$ then $n \leq x$ so $x \in S$. Since $S$ is bounded above by $x$. By the well-ordering property of integers, there exists a greatest element in $S$ denoted as $n$. Since $n$ is the greatest integer less than $x$, we have $n \leq x < n+1$. Therefore, we have shown that for any real number $x$, there exists an integer $n$ such that $n \leq x < n + 1$.

- 2) $\Longrightarrow$ 3). Given $x, y \in \mathbb{R} : x < y$. Due to 2) there exists $q \in \mathbb{Z}^*$ such that

$$q - 1 \leq \frac{1}{y-x} < q.$$

Which implies that

$$1 < q(y - x)$$

Then

$$qx + 1 < qy$$

By 2), there exists $p \in \mathbb{Z}$ such that $p - 1 \leq qx < p$. Hence

$$qx < p \leq qx + 1 < qy$$

Consequently, dividing by $q$, it follows $\boxed{x < \frac{p}{q} < y}$.

- 3) $\Longrightarrow$ 1). Given $x \in \mathbb{R}_+^*$, $y \in \mathbb{R}$. If $x \geq y$ it is enough to take $n = 1$. If not then $0 < x < y$. from 3), there are $p, q \in \mathbb{N}^*$ such that $\frac{p}{q} \geq \frac{y}{x}$ and then $px \geq qy \geq y$, $(q \geq 1)$.

$\square$

**Corollary 1.6.** the irrational set $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

*Proof.* Given $x, y \in \mathbb{R}$ such that $x < y$. form the density of $\mathbb{Q}$, there are $r_1, r_2 \in \mathbb{Q}$ such that $x < r_1 < r_2 < y$. We know that $\sqrt{2}$ is irrational and greater than 1. Then taking $z = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1) \notin \mathbb{Q}$ we obtain $r_1 < z < r_2$. $\square$