Sets, Relations, and Functions

2.1 Set Theory

A set is a collection of objects called **elements** or **members**. The elements in a set can be any types of objects. The members of a set do not even have to be of the same type. Set can be finite or infinite.

$A = \{1, 2, 4, 6, 8, 9\}$. $\mathbb{Z}_+ = \{1, 2, 3, ...\}$, ...

Let $A = \{1, 2, \text{red}\}$. This is read, " A is the set containing the elements 1, 2 and red. We use curly braces " $\{,\}$" to enclose elements of a set.

Special sets

$\emptyset$ or $\{\}$ The **empty** (or **void**, or **null**) set is the set which contains **no elements**.

U : The **universe** set is the set of all elements.

$\mathbb{N}$: The set of **natural** numbers. That is, $\mathbb{N} = \{0, 1, 2, 3, , , , \}$.

$\mathbb{Z}$: The set of **integers**. That is, $\mathbb{Z} = \{., ., ., -2, -1, 0, 1, 2, ., ., .\}$.

$\mathbb{Q}$: The set of **rational** numbers, $\mathbb{Q} = \left\{x \mid x = \frac{a}{b}, (a \in \mathbb{Z},\ b \in \mathbb{Z}^*)\right\}$.

$\mathbb{R}$: The set of **real** numbers.

$\mathbb{C}$: The set of **complex** numbers.

$\rho(A)$: The **power set** of any set A is the set of all subsets of A.

Let $A = \{1, 2\}$. The subsets of A are: $\emptyset, \{1\}, \{2\}$ and $\{1, 2\}$.

Therefore, $\rho(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Set Theory Notation

$\{,\}$: set.

$\in$: $x \in A$: x **is an element** of the set A or x **belongs to** A.

$\notin$: $x \notin A$: x **is not an element** of A.

$\subset$: $A \subset B$: A is a **proper subset** of B.

$\subseteq$: $A \subseteq B$: A is a **subset** of B or B is the **superset** of A.

$=$: $A = B$: **Equal** sets.

$\cap$: $A \cap B$: A **intersection** of B .

$\cup$: $A \cup B$: A **union** of B.

$\times$: $A \times B$ is the **Cartesian product** of A and B.

$\setminus$: $A \setminus B$ is the **difference** of A and B.

$\overline{A}$: is the **complement** of A.

Cardinality of Sets

A is said to be finite if it has a finite number of elements. The number of elements in a finite set A is called its **cardinality (**or **size)**, and is denoted by $|A|$ or $n(A)$.

Hence, $|A|$ is always non negative. If A is an infinite set, some authors would write $|A| = \infty$.

Examples

Let $A = \{1, 3, 7.8.9\}$. Then $|A| = 5$.

$B = \{1, \{2, 3, 4\}, \emptyset\}$. Then $|B| = 3$.

$C = \mathbb{Z}_+ = \{1, 2, 3, ...\}$. Then $|C| = \infty$.

Definition: Subset, proper subset, and Equality

Let A and B be sets.

•A is a **subset** of B, (denoted $A \subseteq B$), if all elements of A are also elements of B. The relation "$\subseteq$" is called the inclusion relation.

$(A \subseteq B) \iff (\forall x \in A \Longrightarrow x \in B)$.

•A is a **proper subset** of B (denoted $A \subset B$) if $A \subseteq B$ and $A \neq B$.

•A is **equal to** B, denoted $A = B$, if $A \subseteq B$ and $B \subseteq A$.

$(A = B) \iff (\forall x \in A, x \in B$ and $\forall x \in B, x \in A)$.

Examples

1) $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

2) The set $\{1, 2\}$ is a proper subset of the set $\{1, 2, 3\}$.

3) $A = \{2, 3, 4, 5\}$, $B = \{2, 3, 4\}$, $C = \{2, 3, 4, 5\}$.

$B \subseteq A$, $B \subset A$ and $C \subseteq A$.

4) $\{1, 2, 7\} \subseteq \{1, 2, 3, 6, 7, 9\}$, but $\{1, 2, 7\} \nsubseteq \{1, 2.3.6.8.9\}$.

2.1.1 Operations on sets

Définition 2.1.1 *:* $A \cap B$

The **intersection** of two sets A and B is the set containing all elements that are in both A and B .

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$
$$(x \in A \cap B) \Longleftrightarrow (x \in A \wedge x \in B).$$
$$(x \notin A \cap B) \Longleftrightarrow (x \notin A \vee x \notin B).$$

If $A \cap B = \emptyset$, so A and B are disjoint.

Définition 2.1.2 *:* $A \cup B$

The **union** of sets A and B is the set containing all elements which are elements of A or B or both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$
$$(x \in A \cup B) \Longleftrightarrow (x \in A \vee x \in B).$$
$$(x \notin A \cup B) \Longleftrightarrow (x \notin A \wedge x \notin B).$$

Examples

1) Let $A = \{0, 1\}$ and $B = \{1, 2, 3\}$.

- What is $A \cup B$?. $A \cup B = \{0, 1, 2, 3\}$.

- What is $A \cap B$?. $A \cap B = \{1\}$.

2) $A = \{x \in \mathbb{N} \mid x \text{ is odd}\}$ and $B = \{x \in \mathbb{N} \mid x \text{ is even}\}$. $A \cup B = \mathbb{N}$, and $A \cap B = \emptyset$.

3)Write, in interval notation, $[5, 8[\, \cup \,]6, 9]$ and $[5, 8[\, \cap \,]6, 9]$.

$[5, 8[\, \cup \,]6, 9] = [5, 9]$, and $[5, 8[\, \cap \,]6, 9] = \,]6, 8[$.

Propositions: Let A, B, and C be three sets. We have:

1) $\emptyset \subset A$ and $A \subset A$.

2) $A \subset (A \cup B)$ and $B \subset (A \cup B)$.

3) $(A \cap B) \subset A$, and $(A \cap B) \subset B$.

4) $(A \cap B) \subset (A \cup B)$.

8) $A \cap \emptyset = \emptyset$, and $A \cap A = A$.

9) $A \cup (B \cup C) = (A \cup B) \cup C$.

10) $A \cap (B \cap C) = (A \cap B) \cap C$.

11) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

5) $A \cup \emptyset = A$, and $A \cup A = A$.

6) if $A \subseteq B$, then $A \cup B = B \cup A = B$.

7) if $A \subseteq B$, then $A \cap B = B \cap A = A$.

12) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Let A and B be two sets in a univers U .

Définition 2.1.3 : A−B

The set difference $A - B$, sometimes written as $A \setminus B$ is the set containing all elements of A which are not elements of B .

$$A \setminus B = \{x \in U \mid x \in A \wedge x \notin B\}.$$

Définition 2.1.4 : $A \triangle B$

The symmetric difference $A \triangle B$, is defined as :

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

Définition 2.1.5 : $\overline{A}$

The complement of A , denoted by $\overline{A}$, $A^c, C_U(A)$, is defined as $\overline{A} = U \setminus A = \{x \in U \mid x \notin A\}$.

Example

Let $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, and $B = \{3, 4\}$.

Find $A \cap B$, $A \cup B$, $A \setminus B$, $B \setminus A$, $A \triangle B$, $\overline{A}$, $\overline{B}$.

Solution

We have:

$A \cap B = \{3\}$.

$A \cup B = \{1, 2, 3, 4\}$.

$A \setminus B = \{1, 2\}$.

$B \setminus A = \{4\}$.

$A \triangle B = \{1, 2, 4\}$.

$\overline{A} = \{4, 5\}$, and $\overline{B} = \{1, 2, 5\}$.

Propositions

1) $A \setminus A = \emptyset$, and $A \setminus \emptyset = A$.

2) $A \cup \overline{A} = U$, and $A \cap \overline{A} = \emptyset$.

3) $\overline{\overline{A}} = A$.

4) $\overline{A \cap B} = \overline{A} \cup \overline{B}$, and $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (De Morgan's laws).

5) if $A \subset B$, then $\overline{B} \subset \overline{A}$.

6)$A \setminus B = A \cap \overline{B}$ and $\overline{(A \setminus B)} = \overline{A} \cup B$.

Exercise

Prove the propositions (4) and (5).

We prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$$\begin{aligned}
\text{Let } x \in \overline{A \cap B} &\Longleftrightarrow x \notin A \cap B. \\
&\Longleftrightarrow \overline{x \in A \cap B}. \\
&\Longleftrightarrow \overline{x \in A \text{ and } x \in B}. \\
&\Longleftrightarrow \overline{x \in A} \text{ or } \overline{x \in B}. \\
&\Longleftrightarrow x \notin A \text{ or } x \notin B. \\
&\Longleftrightarrow x \in \overline{A} \text{ or } x \in \overline{B} \\
&\Longleftrightarrow x \in \overline{A} \cup \overline{B}.
\end{aligned}$$

Définition 2.1.6 *:* $A \times B$

The **Cartesian product** of A and B is the set $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$.

Thus, $A \times B$ (read as "A cross B") contains all the ordered pairs in which the first elements are selected from A, and the second elements are selected from B.

We denoted $A^2 = A \times A$.

Example

1) $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$.

2) Let $A = \{1, 2\}$, and $B = \{2, 5, 6\}$. Then $A \times B = \{(1, 2), (1, 5), (1, 6), (2, 2), (2, 5), (2, 6)\}$.

2.2 Binary relation

Let X and Y be two sets. A binary relation $\Re$ from X to Y is a subset $\Re \subseteq X \times Y$.

If $\Re$ is a relation between X and Y and $(x, y) \in \Re$, we say x is related to y by $\Re$. We write $x \Re y$.

If $\Re$ is a relation from X to X, then we say $\Re$ is a relation on set X.

Examples

1) Let $A = \{0, 1, 2\}$ and $B = \{a, c\}$.

$\{(0, a), (0, c), (1, a), (2, c)\}$ is a binary relation frome A to B.

$\{(0, 0), (0, 2), (1, 2)\}$ is a binary relation on A.

2) We can define a relation $\Re$ on the set of positive integers such that $x \Re y$ if and only if $x \mid y$.

$(x \Re y \Longleftrightarrow x \mid y)$. $\Re = \{(2, 4), (3, 6), (1, 5), (2, 8), ...\}$.

$3\Re$ 6. But 13 is not related to 6 by $\Re$.

3) We can define a relation $\Re$ on the set of real numbers such that $a \Re b$ if and only if $a > b + 1$. $(a \Re b \Longleftrightarrow a > b + 1)$.

2 is not related to 3. $(2 > 3 + 1)$ is false.

5 is related to 3. because $5 > 3 + 1$.

Définition 2.2.1 : *Let $\Re$ be a binary relation on X. We say that $\Re$ is:*

1) **reflexive** if: $\forall x \in X : x \Re x$.

- "=" is reflexive because $x = x$ for any x.
- "$\subseteq$" is reflexive because $A \subseteq A$ for any set A.
- "$\leq$ " is reflexive, but "$<$ " is not reflexive, because $x \nless x$.

2) **symmetric** if: $\forall x, y \in X$, $x \Re y \Longrightarrow y \Re x$.

"=" is symmetric: $x = y \Longrightarrow y = x$ for any x and y.

"$\subseteq$" is not symmetric) because $A \subseteq B \nRightarrow B \subseteq A$.

3) **antisymmetric** if: $\forall x, y \in X$, $(x \Re y$ and $y \Re x) \Longrightarrow x = y$.

4) **transitive** if : $\forall x, y, z \in X$, $(x \Re y$ and $y \Re z) \Longrightarrow x \Re z$.

Exercise 1: Is the relation $\Re$ defined on $\mathbb{Z}$ by:

$$x \Re y \Longleftrightarrow x = -y.$$

reflexive? Is it symmetric? Is it anti-symmetric? Is it transitive?

Solution

1) $\Re$ is **not reflexive**: If it were, we would have:

$$\forall x \in \mathbb{Z} : x \Re x.$$

i.e.

$$\forall x \in \mathbb{Z} : x = -x.$$

But $\exists x = 1 \in \mathbb{Z}$ such that $1 = x$, $-x = -1.(1 \neq -1)$.

Hence, $\Re$ is not reflexive.

2) $\Re$ is **symmetric** because for all $x, y \in \mathbb{Z}$:

$x \Re y \Longleftrightarrow x = -y \Longrightarrow y = -x \Longleftrightarrow y \Re x.$

3) $\Re$ is **not anti-symmetric** because: $\exists 1, -1 \in \mathbb{Z} : 1\Re(-1)$ and $(-1)\Re 1$, but $1 \neq -1$.

4) $\Re$ is **not transitive**: For example, $\exists 1, -1 \in \mathbb{Z} : 1\Re(-1)$ and $(-1)\Re 1$, but 1 is not related to 1 by $\Re$.

Exercise 2

We can define a relation $\Re$ on the set of positive integers such that $x \Re y$ if and only if $x \mid y$. ($x \Re y \Longleftrightarrow x \mid y$).

- This relation is reflexive because $x \mid x$ for all x.
- " $\mid$ " is **NOT symmetric** because, $\exists 2,\ 4 \in \mathbb{Z} : 2 \Re 4$ but 4 is not related to 2 by $\Re$. $(2 \mid 4$, but $4 \nmid 2)$
- This relation is **anti-symmetric** because $x \mid y$ and $y \mid x$ implies that $x = y$.
- This relation is **transitive** because $x \mid y$ and $y \mid z$ implies that $x \mid z$.

2.2.1 Equivalence relation

Définition 2.2.2

An equivalence relation is a relation that is **reflexive**, **symmetric** and **transitive.**

Définition 2.2.3 ***(Equivalence Classes)***

Let $\Re$ be an equivalence relation on X. The equivalence class of $x \in X$, denoted by $\overline{x}$ (or $\overset{\bullet}{x}$), is defined by:

$$\overline{x} = \overset{\bullet}{x} = \{y \in X : x \ \Re \ y\}.$$

$\overline{x}$ is the set of all elements of X that are related to x.

The collection of all equivalent classes of X, denoted by $X \ / \ \Re$ is called the quotient of X by $\Re$, that is,

$$X \ / \ \Re = \left\{ \overset{\bullet}{x} : x \in X \right\}.$$

Propositions

Let $\Re$ be an equivalence relation on X and let $\overset{\bullet}{x}$ be the equivalent class of $x \in X$. Then:

(1) $\forall x \in X : x \in \overset{\bullet}{x}$.

(2) $x \ \Re \ y \Longleftrightarrow \overset{\bullet}{x} = \overset{\bullet}{y}$.

(3) If $\overset{\bullet}{x} \neq \overset{\bullet}{y}$, then $\overset{\bullet}{x}$ and $\overset{\bullet}{y}$ must be disjoint.

Exercise 01

Let $\Re$ be a relations on the set $X = \{4, 5, 6, 7\}$ defined by:

$\Re = \{(4, 4), (5, 5), (6, 6), (7, 7), (4, 6), (6, 4)\}$.

a) Show that $\Re$ is an Equivalence Relation.

b) Determine its equivalence classes.

Solution

a.1) **Reflexive**: Relation $\Re$ is reflexive as for every $x \in X$. $(x, x) \in \Re$, i.e. $(4, 4), (5, 5), (6, 6),$ and $(7, 7) \in \Re$.

a.2) **Symmetric**: Relation $\Re$ is symmetric because whenever $(a, b) \in \Re$;

(b,a) also belongs to $\Re$. Example: $(4,6) \in \Re \Longrightarrow (6,4) \in \Re$.

a.3) **Transitive**: Relation $\Re$ is transitive because whenever (x,y) and (y,z) belongs to $\Re$: (a,c) also belongs to $\Re$.

Example: $(4,6) \in \Re$ and $(6,4) \in \Re \Longrightarrow (4,4) \in \Re$.

As the relation $\Re$ is reflexive, symmetric and transitive. Hence, $\Re$ is an Equivalence Relation.

b) The equivalence classes are as follows:

$\overline{4} = \{4,6\} = \overline{6}$.

$\overline{5} = \{5\}$

$\overline{7} = \{7\}$.

Exercise 02

We define on $\mathbb{Z}$ a relation $\Re$ as follows:

$$x \;\Re\; y \Longleftrightarrow x = y$$

Show that $\Re$ is an equivalence relation.

Solution

This relation is **reflexive** because $\forall x \in \mathbb{Z} : x = x \Longrightarrow x \;\Re\; x$.

2) $\Re$ is **symmetric** because for all $x, y \in \mathbb{Z}$:

$$x \;\Re\; y \Longleftrightarrow x = y \Longrightarrow y = x \Longrightarrow y \;\Re\; x.$$

3) $\Re$ is **transitive** because for all $x, y, z \in \mathbb{Z}$:

$$\begin{cases} x\Re y \Longleftrightarrow x = y \\ \qquad\quad \wedge \\ y\Re z \Longleftrightarrow y = z \end{cases} \Longrightarrow x = z \Longrightarrow x \;\Re\; z.$$

Thus, $\Re$ is an equivalence relation.

Exercise 03

"divides": ($x \, \Re \, y \Longleftrightarrow x \mid y$) is not an equivalence relation. Because is not symmetric. $\exists 2,\, 4 \in \mathbb{Z}_+ : 2 \, \Re \, 4$ but 4 is not related to 2 by $\Re$.

Exercise 04

We define on $\mathbb{Z}$ a relation $\Re$ as follows:

$$x \, \Re \, y \Longleftrightarrow x + y \text{ is even.}$$

a) Show that $\Re$ is an equivalence relation.

b) What are the equivalence classes of 0 and 1?

Solution

a.1) Let $x \in \mathbb{Z}$. Since $x + x = 2x$ is always even, $\Re$ is **reflexive**.

a.2) Let $x, y \in \mathbb{Z}$. $x + y = y + x$, $x + y$ is even if and only if $y + x$ is so. Thus $\Re$ is **symmetric**.

a.3) The relation $\Re$ is **transitive**. To prove this, let $x, y, z \in \mathbb{Z}$, and assume that $x \, \Re \, y$ and $y \, \Re \, z$, i.e. $x + y$ and $y + z$ are even. So, there exist $n, m \in \mathbb{Z}$ such that $x + y = 2n$ and $y + z = 2m$.

Thus, $x + y + y + z = 2n + 2m \Longrightarrow x + z = 2(n + m - y)$

i.e. $x + z$ is even, that is, $x \, \Re \, z$. ($\Re$ is transitive).

Therefore $\Re$ is is an equivalence relation.

b) equivalence classes of 0 and 1:

$\overline{0} = \{y \in \mathbb{Z} : 0 \, \Re \, y\} = \{y \in \mathbb{Z} : 0 + y \text{ is even}\} = \{0, \pm 2, \pm 4, ...\}$.

$\overline{1} = \{y \in \mathbb{Z} : 1 \, \Re \, y\} = \{y \in \mathbb{Z} : 1 + y \text{ is even}\} = \{\pm 1, \pm 3, \pm 5, ...\}$.

$\overline{2} = \{y \in \mathbb{Z} : 2 \, \Re \, y\} = \{y \in \mathbb{Z} : 2 + y \text{ is even}\} = \{0, \pm 2, \pm 4, ...\}$.($\overline{0} = \overline{2}$, because $0 \, \Re \, 2$.)

$\overline{0}$ and $\overline{1}$ are the only equivalence classes withe respecte to this equivalence relation.

2.2.2 Order relation

Définition 2.2.4 ***Partial order, total order***

A relation $\Re$ on a set X is called a partial order relation if it satisfies the following three properties:

Relation $\Re$ is **Reflexive**, i.e. $\forall x \in X : x \Re x$.

Relation R is **Antisymmetric**, i.e. $\forall x, y \in X$, $(x \Re y \text{ and } y \Re x) \Longrightarrow x = y$.

Relation R is **transitive**, i.e. $\forall x, y, z \in X$, $(x \Re y \text{ and } y \Re z) \Longrightarrow x \Re z$.

A partial order is said to be a **total** order if for any $x, y \in X$ either $x \Re y$ or $y \Re x$.

A pair $(X, \Re)$, where $\Re$ is a partial order over X, is called a partial order set or **poset.**

1) $(\mathbb{Z}; \leq)$ is a total order set.

2) $(\mathbb{Z}; |)$ is a partial ordere set but not total order set.

Exercise 01

Show that the "greater than or equal" relation $(\geq)$ is a partial order on the set of integers.

$$(x\Re y \Longleftrightarrow x \geq y)$$

Solution:

1) **Reflexivity**: $x \geq x$ for every integer x.

2) **Antisymmetry**: If $x \geq y$ and $y \geq x$; then $x = y$.

3) **Transitivity**: If $x \geq y$ and $y \geq z$; then $x \geq z$.

These properties all follow from the order axioms for the integers.

Exercise 02

Show that the relation **divides** defined on $\mathbb{N}$ is a partial order relation.

Solution:

1) **Reflexivity**: We have x divides x, $\forall x \in \mathbb{Z}_+$. Therefore, relation "Divides" is reflexive.

2) **Antisymmetry**: If x and y are positive integers with $x \mid y$ and $y \mid x$; then $x = y$.

3) **Transitivity**: Suppose that x divides y and that y divides z. Then, there are positive integers k and l such that $y = xk$ and $z = yl$, $z = x(kl\ )$, so that x divides z.

Hence the relation is transitive. Therefore, the relation **divides** is a partial order on the set of positive integers.

Exercise 03

Show that the **inclusion** relation "$\subseteq$" is a partial order on the power set of a set S.

Solution:

1) **Reflexivity**: $A \subseteq A$ whenever A is a subset of S.

2) **Antisymmetry**: If A and B are subsets of S, with $A \subseteq B$ and $B \subseteq A$; then $A = B$.

3) **Transitivity**: If $A \subseteq B$ and $B \subseteq C$; then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Upper Bounds, Lower Bounds, Sup, Inf

Let $(X, \leq)$ be a partially ordered set, and let A be a subset of X.

Définition 2.2.5 ***(upper bound)***

$u \in X$ is an **upper bound** or **majorant** of A if every element of A is less then or equal to u. i.e. $u \geq x$ for all $x \in A$.

if A has an upper bound, then we say that A is **bounded above.**

Note that the upper bounds don't need to belong to the subset).

Example

$A = [-1, 3[\subset \mathbb{R}$. $u = 3$ is an upper bound of A.(any real number $u' \geq 3$ is also an upper bound of A.

A is bounded above.

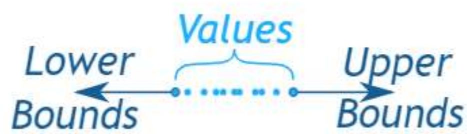
2) let $A = \mathbb{N} = \{0, 1, 2, ...\}$ .A does not have any upper bound. Then A is not bounded above.

Définition 2.2.6 ***(lower bound)***

$l \in X$ is a **lower bound** or **minorant** of A if every element of A is greater then or equal to l. i.e. $l \leq x$ for all $x \in A$.

if A has an lower bound, then we say that A is **bounded below.**

Note that the lower bounds don't need to belong to the subset.



Example

$A = [-1, 3[\subset \mathbb{R}$. $l = -1$ is a lower bound of A.(any real number $l' \leq -1$ is also a lower bound of A. (A is bounded below).

($l = -1.5$ is a lower bound of A, but $l = -0.5$ is NOT a lower bound of A.

Définition 2.2.7 ***(bounded sets)***

we say that A is **bounded** If it is both bounded above and below.

Examples

1) $A = [-1, 3[\subset \mathbb{R}$ is bounded.

2) The set of natural numbers, i.e. $\mathbb{N} = \{0, 1, 2, 3, ...\}$ is a set which is bounded below (by 0), but not bounded above.($\mathbb{N}$ is not bounded)

3) $B = \{1, -2, 7\} \subset \mathbb{Z}$. Then A is bounded above (e.g. , by $7, 8, 10, ...$) and below (e.g. ,by $-2, -3, -8, ...$)

4) Let $C = \{1, 2\}$ be a subset of the set of natural numbers $\mathbb{N}$, then $2, 3, 4, 5, ...$ will all be upper bounds of C (C is bounded above), and $0, 1$ will be lower bounds of C (C is bounded below). Then we say that C is bounded.

5) Consider $D =]0, 1]$ of $\mathbb{R}$. Any real number greater than or equal to 1 is an upper bound of D, and any real number less than or equal to 0 is a lower bound of D. (D is bounded).

Définition 2.2.8 *(supremum infimum)*

Let X be a partially ordered set, and let A be a subset of X.

1) An element $u_0 \in X$ is a "**least upper bound**" or "**supremum"** of A if it is smallest of all upper bounds u.

If a supremum exists, it is denoted by $\sup(A)$.

$\sup(A)$ may or may not belongs to set A.

2) An element $l_0 \in X$ is a "greatest lower bound" or **infimum** of A if it is greatest of all lower bounds l.

If an infimum exists, it is denoted by $\inf(A)$.

$\inf(A)$ may or may not belongs to set A.

Remark

If $u_0 \in A$.We say that u_0 is the **maximum** (greatest element) of A and write

$$u_0 = \max(A) = \sup(A).$$

If $l_0 \in A$.We say that l_0 is the **minimum** (smallest element) of A and write

$$l_0 = \min(A) = \inf(A).$$

Example

1) $]0, 1]$ is a subset of $\mathbb{R}$. The set of all upper bounds of A is the set $B = [1, +\infty[$, $\sup(A) = 1$. And $1 \in A$: 1 is the maximum of A. *i.e.* $\max(A) = 1$.

The set of all lower bounds of A is the set $C =]-\infty, 0]$, $\inf(A) = 0$. But $\min A$ does not exists because $\inf(A) \notin A$.

2) $S = \{1, 2, 3, 4\} \subset \mathbb{N}$, then $\sup(S) = \max(S) = 4$, because $4 \in S$ and every $s \in S$ satisfies $s \leq 4$.

and $\inf(S) = \min(S) = 1$.

3)$S = [0, 1]$, then $\sup(S) = \max(S) = 1$, and $\inf(S) = \min(S) = 0$.

Propositions

1)The supremum or infimum of a set A is unique if it exists.

2) If A, B are nonempty sets, then $\sup(A \pm B) = \sup(A) \pm \sup(B)$, $\inf(A \pm B) = \inf(A) \pm \inf(B)$.

Exercise

Find the sup, inf, max, and min of the following set.

$A = \left\{\frac{1}{n},\ n \in \mathbb{N}^*\right\}$

Solution

We write the first few terms of S:

$$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$$

Then $\sup(A) = 1$ belongs to A, so $\max(A) = \sup(A) = 1$. On the other hand, $\inf(A) = 0$ doesn't belong to A, ($\lim_{n\to\infty} \frac{1}{n} = 0$). So A has no minimum.

Exercise Let $(E, \leq)$ be an ordered set, and let A be a subset of E.

Find the sup, inf, max, and min of the following sets, if it exists.

1) $E = \mathbb{R}$, $A = \{0, 1, -5, 3, 5, -2\}$.

2) $E = \mathbb{R}$, $A = [-4, 2[$.

3) $E = \mathbb{R}$, $A =]-1, 1[$.

4) $E = \mathbb{R}$, $A =]-\infty, 2[$.

5) $E = [-1, 1]$, $A = \left\{\cos(\frac{7\pi n}{2},\ n \in \mathbb{Z}\right\}$.

6) $E = \mathbb{R}$, $A = \{x^2 - 1,\ x \in \mathbb{R}\}$.

2.3 Functions

A function $f : X \rightarrow Y$ is a rule that, for every element $x \in X$, associates an element $f(x) \in Y$. The element $f(x)$ is sometimes called the image of x, and the subset of Y consisting of images of elements in X is called the image of f. That is,

$$image(f) = \{y \in Y : \ y = f(x) \text{ for some } x \in X\}$$

Is a function always a relation?

A function is always a relation. A function is a type of relation in which each input has a unique output, mning an input does not have more than one output.

A relation is not a function if there is more than one output for an input. For example, in the relation $\{(1,0), (1,2), (2,3)\}$, the input of 1 gives two different outputs. So the relation is not a function.

Définition 2.3.1 ***(Image, Pre-image, Domain and Range of a Function)***

Domain and **co-domain**: if f is a function from set X to set Y, then X is called **Domain** and Y is called **co-domain**.

$$D(f) = \{x : x \in A \text{ for which } f(x) \text{ is defined}\}$$

Image and Pre-Image: If $y \in Y$ is associated with an element $x \in X$, we write it as, $y = f(x)$

which is read "y equals f of x".$f\ (x)$ is known as the **image** of f at x or value of f at x. and x is called the pre-image of y .

Range: Range of f is the set of all images of elements of X. Basically Range is subset of co- domain. $(R(f) \subseteq Y)$

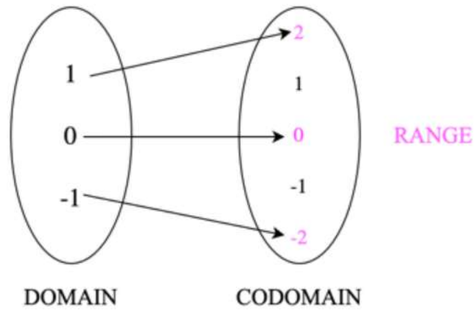
$$R(f) = \{y : y \in Y, \ y = f(x) \text{ for all } x \in X\}.$$

Examples

$y = f(x) = \sqrt{x-1}$.Then $f(x)$ is defined for $x - 1 \geq 0$ i.e. $x \geq 1$. Thus,

$$D(f) = \{x : x \geq 1\} = [1, \infty[.$$

$$R(f) = \{y : y \geq 0\} = [0, \infty[.$$



2.3.1 Properties of Function

Addition and multiplication: let f and g are two functions from X to Y, then $f + g$ and $f \cdot g$ are defined as:

$f + g(x) = f(x) + g(x)$. (addition)

$fg(x) = f(x)g(x)$. (multiplication)

Equality: If two functions f and g :$X \to Y$ have a same domain , then they are said to be equal iff $f(x) = g(x)$ for every $x \in X$ and is written as $f = g$.

Composition: If $f : A \to B$ and $g : B \to C$ be any two functions, then the composite function of f and g, denoted by $g \circ f$ (read as "g of f") is the function $g \circ f : A \to C$ and defined by the equation,

$$(g \circ f)(x) = g(f(x)).$$

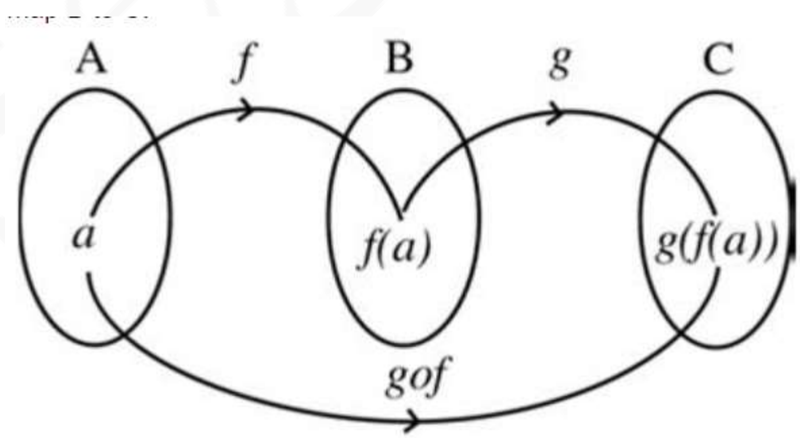


Figure 2.3.1 : composition of f and g

Example

Let A, B and C denote the sets of real numbers. Suppose $f : A \to B$ and $g : B \to C$ are defined by

$$f(x) = x - 1; \quad g(x) = x^2$$

Then,

$$(\ g \circ f\ )(x) = g(f(x)) = g(x-1).$$
$$= (x-1)^2.$$

2.3.2 Direct image, inverse image

Définition 2.3.2 *(direct image of a set)*

Let $f : X \to Y$ and $A \subset X$, the direct image of A under the function f written $f(A)$ is the set

$$f(A) = \{f(x) : x \in A\}$$

Définition 2.3.3 *(Inverse image, pre-image of an element)*

Let $f : X \to Y$ and $b \in Y$. Then the inverse image of b under f, $f^{-1}(b)$, is the set

$$f^{-1}(b) = \{x \in X : f(x) = b\}.$$

Définition 2.3.4 *(Inverse image, pre-image of a subset)*

Let $f : X \to Y$ is a function where $B \subset Y$ then the inverse image of B under the function f is the set:

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Examples

Let f be as in Figure 2.3.2

Then $f(\{b, c\}) = \{1, 3\}$, $f^{-1}(1) = \{a, b\}$, and $f^{-1}(\{1, 3\}) = \{a, b, c, d\}$.

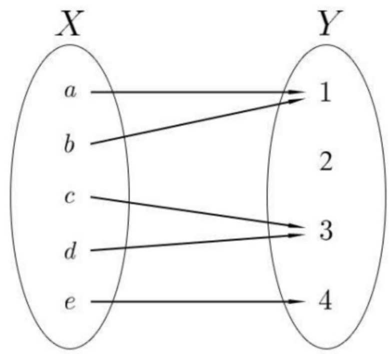


Figure 2.3.2 : Picture of f

Exercise

Let $E = [0, 1]$ and $F = [-1, 0]$ be two intervals of $\mathbb{R}$.

We consider a function $f : E \to F$, defined by $f(x) = x^2 - 1$.

Determine $f(\left[0, \frac{1}{2}\right[)$, $f^{-1}(-\frac{1}{2})$, and $f^{-1}(\left]\frac{-1}{2}, 0\right[$.

Solution

1) $f(\left[0, \frac{1}{2}\right[) = \left\{f(x) \in F, x \in \left[0, \frac{1}{2}\right[\right\}$.

$x \in \left[0, \frac{1}{2}\right[\Longrightarrow 0 \leq x < \frac{1}{2} \Longrightarrow 0 \leq x^2 < \frac{1}{4}$.

$$\Longrightarrow -1 \leq x^2 - 1 < -\frac{3}{4}.$$

$f(\left[0, \frac{1}{2}\right[) = \left[-1, -\frac{3}{4}\right[$.

2) $f^{-1}(-\frac{1}{2}) = \left\{x \in [0, 1], f(x) = -\frac{1}{2}\right\}$.

$f(x) = -\frac{1}{2} \Longrightarrow x^2 - 1 = -\frac{1}{2} \Longrightarrow x^2 = \frac{1}{2}$.

$$\Longrightarrow x = \frac{1}{\sqrt{2}} (\text{because } x \text{ is a positive number}).$$

$f^{-1}(-\frac{1}{2}) = \frac{1}{\sqrt{2}}$.

3) $f^{-1}(\left]\frac{-1}{2}, 0\right[= \left\{x \in [0, 1], f(x) \in \left]-\frac{1}{2}, 0\right[\right\}$.

$f(x) \in \left]-\frac{1}{2}, 0\right[\Longrightarrow -\frac{1}{2} < x^2 - 1 < 0 \Longrightarrow \frac{1}{2} < x^2 < 1$.

$$\Longrightarrow \frac{1}{\sqrt{2}} < |x| < 1 \Longrightarrow \begin{cases} \frac{1}{\sqrt{2}} < x < 1 \\ -1 < x < -\frac{1}{\sqrt{2}} \end{cases}.$$

But $x \in [0, 1]$, then $f^{-1}(\left]\frac{-1}{2}, 0\right[= \left]\frac{1}{\sqrt{2}}, 1\right[$.

2.3.3 Types of functions: injective, surjective and bijective

Injective function or (one-to-one)

Définition 2.3.5

Let $f : X \to Y$ be a function. Then f is **injective** or "**one -to-one**" if for all elements x_1 and $x_2$in X , if $f(x_1) = f(x_2)$, then it must be the case that $x_1 = x_2$.

This is equivalent to saying if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.(contrapositive)

If X and Y are finite sets and $f : X \to Y$ is injective, then $|X| \leq |Y|$.

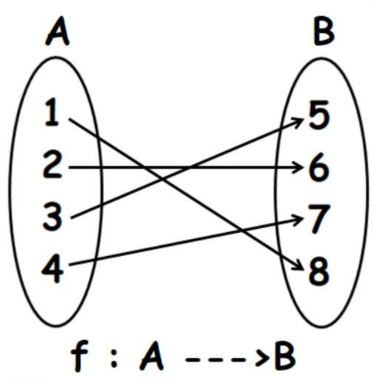


Figure 2.3.3 : injective

Example

The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$ is injective if :

$f(x_1) = f(x_2) \Longrightarrow 2x_1 = 2x_2$, dividing both sides by 2 yields $x_1 = x_2$.

Surjective function or (onto)

Let $f : X \to Y$ be a function. If every element of Y is the image of at least one element of X. i.e. every element of Y has a pre-image, thenThat is, $f(X) = Y$.

Symbolically,

$$\forall y \in Y, \exists x \in X \text{ such that } f(x) = y.$$

If X and Y are finite sets and $f : X \to Y$ is surjective, then $|X| \geq |Y|$.

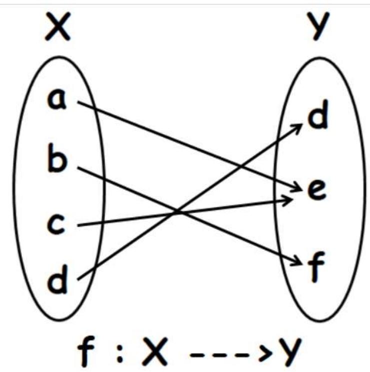


Figure 2.3.4 : Surjective

Example: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$ is not surjective: there is no integer x such that $f(x) = 3$,because

$2x = 3$ has no solutions in $\mathbb{Z}$. So 3 is not in the image of f.

Bijective "one-to-one and onto"

Let $f : X \to Y$ be a function. Then f is **bijective** or (**one-to-one correspondence**) if it is **injective** and **surjective**; that is, every element $y \in Y$ is the image of exactly one element $x \in X$.

$$\forall y \in Y, \exists! x \in X \text{ such that } f(x) = y.$$

If X and Y are finite sets and $f : X \to Y$ is **bijective**, then $|X| = |Y|$.

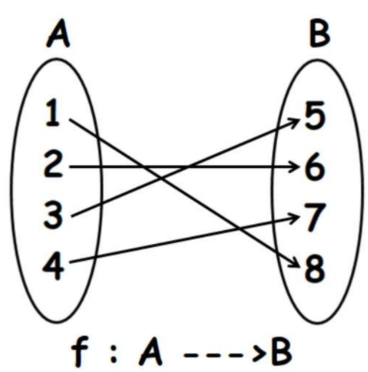


Figure 2.3.5 :

Exercise

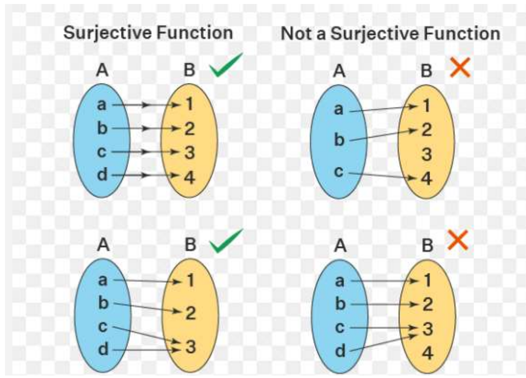


Figure 2.3.6 : Surjective / Not surjective

Inverse functions

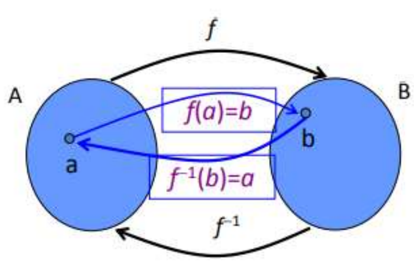
Let $f : A \to B$ be a one-to-one correspondence (bijection). Then the **inverse function of** f , $f^{-1} : B \to A$, associates each element b of B with a unique element a of A such that $f(a) = b$.

$$f^{-1}(b) = a \Longleftrightarrow b = f(a).$$

The inverse is usually shown by putting a little "-1" after the function name, like this: f^{-1}.

Définition 2.3.6 ***(Inverse function)***

If $f : A \to B$ and $g : B \to A$ are functions, we say g is an inverse to f (and f is an inverse to g) if and only if: $f \circ g = I_B$ and $g \circ f = I_A$.



Remark

- $f^{-1}(y)$ is not $\frac{1}{f(y)}$

-$(f^{-1)-1} = f$

Properties

A function $f : A \to B$ has an inverse if and only if it is bijective.

If $f : A \to B$ has an inverse function then the inverse is unique.

The inverse of a bijective function is also a bijection.

The composition of two bijections is a bijection.

Example

Let f be the real function $f(x) = x^2$. The function f is not a bijection, so it does not have an inverse function. However the function

$$g : [0, \infty[\to [0, \infty[$$
$$x \mapsto x^2$$

is a bijection. In this case, $g^{-1}(y) = \sqrt{y}$.

Bijection theorem

$f : I \subset \mathbb{R} \to \mathbb{R}$

If f is continuous and strictly monotonic on I. Then:

1) $f : I \to J = f(I)$ is a bijective.

2) f^{-1} is continuous and strictly monotonic on J, with the same direction of variation as f.

Exercise

Let $f :]0, \infty[\to]0, 1[$ be the function defined by $f(x) = \dfrac{1}{\sqrt{x+1}}$.

1) Determine $f(]2, 4])$ and $f^{-1}(\left]\frac{1}{2}, \frac{\sqrt{3}}{2}\right])$.

2) Show that the function f is bijective and determine f^{-1}.

Solution

1) * $f(]2,4]) = \{y \in]0,1[\,, x \in]2,4]\}$.
$x \in]2,4] \Longrightarrow 2 < x \leq 4 \Longrightarrow \sqrt{3} < \sqrt{x+1} \leq \sqrt{5}$

$$\Longrightarrow \frac{1}{\sqrt{5}} \leq \frac{1}{\sqrt{x+1}} < \frac{1}{\sqrt{3}}.\text{ Then } f(]2,4]) = \left[\tfrac{1}{\sqrt{5}}, \tfrac{1}{\sqrt{3}}\right[.$$

* $f^{-1}(\left]\frac{1}{2}, \frac{\sqrt{3}}{2}\right]) = \left\{x \in]0,\infty[\,, f(x) \in \left]\frac{1}{2}, \frac{\sqrt{3}}{2}\right]\right\}$

$$f(x) \in \left]\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right] \Longrightarrow \frac{1}{2} < \frac{1}{\sqrt{x+1}} \leq \frac{\sqrt{3}}{2} \Longrightarrow \frac{1}{3} \leq x < 3.$$

$f^{-1}(\left]\frac{1}{2}, \frac{\sqrt{3}}{2}\right]) = \left[\frac{1}{3}, 3\right[$.

2) Show that the function f is bijective and determine f^{-1}.

We show that f is injective and surjective

a) f is injective: if $f(x_1) = f(x_2)$, then $x_1 = x_2$ for all $x_1, x_2 \in]0,\infty[$

$$f(x_1) = f(x_2) \Longrightarrow \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{x_2+1}} \Longrightarrow x_1 = x_2.\text{ Then } f \text{ is injective}$$

b) f is surjective: $\forall y \in]0,1[\,, \exists x \in]0,\infty[\,$, such that $f(x) = y$.

$$y = f(x) = \frac{1}{\sqrt{x+1}} \Longrightarrow \sqrt{x+1} = \frac{1}{y} \Longrightarrow x = \frac{1}{y^2} - 1$$

Then $\forall y \in]0,1[\,, \exists x = \dfrac{1}{y^2} - 1 \in]0,\infty[$ such that $y = f(x)$, therefore f is surjective.
f is injective and surjective therefore it is bijective, and

$$f^{-1} :]0,1[\to]0,\infty[\to \text{defined by } f^{-1}(y) = \frac{1}{y^2} - 1.$$