Chapter 01 (part 01). Topological spaces

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Introduction to Topology for Second Class License Academic
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Topological space

Definition – Topology

A topology τ on a set X is a collection of subsets of X, called open sets, satisfying the following axioms:

1. $\emptyset$ and X are open sets (i.e. $\emptyset, X \in \tau$).
2. The intersection of two open sets is an open set (i.e. $O_1 \cap O_2 \in \tau$).
3. The union of any number of open sets is an open set (i.e. $\cup_i O_i \in \tau$).

- Condition 2 implies that any finite intersection of open sets is still an open set.
- A topological space (X, τ) consists of a set X and a topology τ.
- i.e. A set equipped with a topology is called a topological space. Its elements are called points.

Examples of topological spaces

- The trivial (indiscrete) topology on a set X is defined as the topology which consists of the subsets $\emptyset$ and X only (i.e. $\tau = \{\emptyset, X\}$).
- The discrete topology on a set X is defined as the topology which consists of all possible subsets of X (i.e. $\tau = \mathcal{P}(X)$).
- In the Euclidean (standard) topology on $\mathbb{R}$, a subset $U \subset \mathbb{R}$ is open if and only if it is the union of open intervals.
- The upper topology on $\mathbb{R}$ is defined as the one having $]-\infty, \alpha[$, with $\alpha \in \mathbb{R} \bigcup \{+\infty\}$ as non-empty open sets.

Closed sets

Definition – Closed set

Let X be a topological space. A subset $Z \subset X$ is called closed if $X - Z$ is open.

Main facts about closed sets

1. $\emptyset$ and X are are closed (i.e. $X - \emptyset = X, X - X = \emptyset \in \tau$).
2. Any intersection of closed sets is closed.
3. The union of two closed sets is closed.
4. If a subset $Z \subset X$ is closed in X, then every sequence of points of Z that converges must converge to a point of Z.

Closure of a set

Definition – Closure

Let X be a topological space and assume that $D \subset X$. The closure $\overline{D}$ of D is defined as the smallest closed set that contains D. It is thus the intersection of all closed sets that contain D.

Main facts about the closure

1. One has $D \subset \overline{D}$ for any set D.
2. If $D \subset Z$, then $\overline{D} \subset \overline{Z}$ as well.
3. The set D is closed if and only if $\overline{D} = D$.
4. The closure of $\overline{D}$ is itself, namely $\overline{\overline{D}} = \overline{D}$.

- The interval $I = [-1, 2)$ has closure $\bar{I} = [-1, 2]$.
- The interval $I = (-1, 2)$ has closure $\bar{I} = [-1, 2]$.

Interior of a set

Definition – Interior

Let X be a topological space and let $D \subset X$. The interior D° of D is defined as the largest open set contained in D. It is thus the union of all open sets contained in D.

Main facts about the interior

1. One has $D^\circ \subset D$ for any set D.
2. If $D \subset Z$, then $D^\circ \subset Z^\circ$ as well.
3. The set D is open if and only if $D^\circ = D$.
4. The interior of D° is itself, namely $D^{\circ\circ} = D$.

- The interval $I = [0, 1]$ has interior $I^\circ = (0, 1)$.
- The interval $I = [0, 1)$ has interior $I^\circ = (0, 1)$.

Boundary of a set

Definition – Boundary

Let X be a topological space and suppose that $D \subset X$t. The boundary of D is defined as the set

$$\partial D = \overline{D} - D^{\circ} = \overline{D} \cap \overline{X - D}$$

For the Euclidean topology on the real line $\mathbb{R}$ we have

- The boundary of $I = [0, 1]$, $\partial I = \bar{I} - I^{\circ} = \bar{I} \cap \overline{\mathbb{R} - I} = \{0, 1\}$.
- The boundary of $I = [0, 1)$, $\partial I = \bar{I} - I^{\circ} = \bar{I} \cap \overline{\mathbb{R} - I} = \{0\}$.
- Notice that a subset and its complement have the same boundary.

Neighbourhoods

Definition – Neighbourhood

Let X be a topological space and let $x \in X$ be an arbitrary point. A neighbourhood of x is simply an open set that contains x.

Theorem - Characterisation of closure/interior/boundary

Let X be a topological space. Assume that $D \subset X$ is a subset.

1. $x \in \overline{D} \iff$ every neighbourhood of x intersects D.
2. $x \in D^\circ \iff$ some neighbourhood of x lies within D.
3. $x \in \partial D \iff$ every neighbourhood of x intersects D and $X - D$.
4. One has $D^\circ \cap \partial D = \emptyset$ and $D^\circ \cup \partial D = \overline{D}$ for evevry D.

Interior, closure and boundary: examples

Set	Interior	Closure	Boundary
$\{3\}$	$\emptyset$	$\{3\}$	$\{3\}$
$[1,4)$	$(1,4)$	$[1,4]$	$\{1,4\}$
$(-1,2)\cup(2,3)$	$(-1,2)\cup(2,3)$	$[-1,3]$	$\{-1,2,3\}$
$[-1,2]\cup\{3\}$	$(-1,2)$	$[-1,2]\cup\{3\}$	$\{-1,2,3\}$
$\mathbb{Z}$	$\emptyset$	$\mathbb{Z}$	$\mathbb{Z}$
$\mathbb{Q}$	$\emptyset$	$\mathbb{R}$	$\mathbb{R}$
$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\emptyset$

Convergence of sequences

Definition – Convergence

Let (X, τ) be a topological space. A sequence $(x_n)_n$ of points of X is said to converge to the point $x \in X$ if, given any open set U that contains x, there exists an integer q such that $x_n \in U$ for all $n \geq q$.

- When a sequence (x_n) converges to a point x, we say that x is the limit of the sequence and we write $x_n \longrightarrow x$ as $n \longrightarrow +\infty$ or simply

$$\lim_{x_n \longrightarrow x} x_n = x$$

.

Theorem - Limits are not necessarily unique

Assume that X has the indiscrete topology and let $x \in X$. Then the constant sequence $x_n = x$ converges to y for every $y \in X$.

Accumulation point vs Limit point

Definition – Limit point

Let X be a topological space and Suppose that $D \subset X$. We say that x is a limit point of D if every neighbourhood of x intersects D at a point other than x.

Theorem - Limit points and closure

Let X be a topological space and let $D \subset X$. If $D^{'}$ is the set of all limit points of D, then the closure of D is

$$\overline{D} = D \cup D^{'}$$

- Every limit of a non-constant sequence is an accumulation point of the sequence.

Limit points

- A set is closed if and only if it contains its limit points.
- A sequence accumulates at x means that x is a limit point. If a sequence converges to x then it also accumulates to x, so x is a limit point; the converse is generally false.
- Intuitively, limit points of D are limits of sequences of points of D.
- Every point of $D = (0, 3)$ is a limit point of D, while $D^{'} = [0, 3]$.
- The set $D = \{\frac{1}{n^2},\ n \in \mathbb{N}\}$ has only one limit point, namely $x = 0$.

Hausdorff Space

Definition – Hausdorff space

Let (X, τ) be a topological space. We say that (X, τ) is Hausdorff if any two distinct points of X have neighbourhoods which do not intersect.

- If a space X endowed with the discrete topology, then X is Hausdorff.
- If a space X equipped with the indiscrete topology and it contains two or more elements, then X is not Hausdorff.

Theorem – Main facts about Hausdorff spaces

1. A convergent sequence in a Hausdorff space has a unique limit.
2. Every subset of a Hausdorff space is Hausdorff.
3. Every finite subset of a Hausdorff space is closed.
4. The product of two Hausdorff spaces is Hausdorff.

Continuous Maps

Definition – Continuity

A function $f\colon X \longrightarrow Y$ between topological spaces is called continuous if $f^{-1}(U)$ is open in X for each set U which is open in Y.

Theorem - Composition of continuous functions

Suppose $f\colon X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous functions between topological spaces. Then the composition $g \circ f\colon X \longrightarrow Z$ is continuous.

Theorem - Continuity and sequences

Let $f: X \longrightarrow Y$ be a continuous function between topological spaces and let (x_n) be a sequence of points of X which converges to $x \in X$. Then the sequence $(f(x_n))_n$ must converge to $f(x)$.

Definition – Homeomorphism

A function $f: X \longrightarrow Y$ between topological spaces is a homeomorphism if f is bijective, continuous and its inverse f^{-1} is continuous.
When such a function exists, we say that X and Y are homeomorphic.

Subspace Topologies

Definition – Subspace topology

Let (X, τ) be a topological space and let $D \subset X$. Then the set

$$\tau' = \{U \cap D,\ U \in \tau\}$$

forms a topology on D which is known as the subspace topology.

Theorem - Inclusion maps are continuous

Let (X, τ) be a topological space and let $D \subset X$. Then the inclusion map $I : D \longrightarrow X$ which is defined by $I(x) = x$ is continuous.

Theorem - Restriction maps are continuous

Let $f : X \longrightarrow Y$ be a continuous function between topological spaces and let $D \subset X$. Then the restriction map $h : D \longrightarrow Y$ which is defined by $h(x) = f(x)$ is continuous. This map is often denoted by $h = f/D$.

Product topology

Definition – Product topology

Given two topological spaces (X, τ) and $(X, \tau^{'})$, we define the product topology on $X \times Y$ as the collection (open sets) of all unions $\cup_{i,j}(O_i \times O_j)$, where each O_i is open in X and each O_j is open in Y.

Theorem - Restriction maps are continuous

Let (X, τ), $(Y, \tau^{'})$, and $(Z, \tau^{''})$ be topological spaces. Then a function $f: Z \longrightarrow X \times Y$ is continuous if and only if its components $p_1 \circ f$ and $p_2 \circ f$ are continuous when the projection map $p_1 : X \times Y \longrightarrow X$ defined by $p_1(x, y) = x$, and the projection map $p_2 : X \times Y \longrightarrow Y$ defined by $p_2(x, y) = y$ are continuous.

Questions are welcome.