Exercice 01 : $E = \{a, b, c, d\}$.

1. We determine the topologies among the following families :
   i) $\tau_1 = \{\emptyset, E, \{a\}, \{c, d\}, \{a, c, d\}\}$:
   $*)$ $\emptyset, E \in \tau_1$,
   $**)$ $\forall \mathcal{O} \in \tau_1 : \emptyset \cap \mathcal{O} = \emptyset \in \tau_1, E \cap \mathcal{O} = \mathcal{O} \in \tau_1$,
   $\{a\} \cap \{c, d\} = \emptyset \in \tau_1,\ \{a\} \cap \{a, c, d\} = \{a\} \in \tau_1, \{c, d\} \cap \{a, c, d\} = \{c, d\} \in \tau_1$
   $***)$ $\forall \mathcal{O} \in \tau_1 : \emptyset \cup \mathcal{O} = \mathcal{O} \in \tau_1, E \cup \mathcal{O} = E \in \tau_1, \{a\} \cup \{c, d\} = \{a, c, d\} \in \tau_1$,
   $\{a\} \cup \{a, c, d\} = \{a, c, d\} \in \tau_1, \{c, d\} \cup \{a, c, d\} = \{a, c, d\} \in \tau_1$,
   So, τ_1 is a topology.
   ii) $\tau_2 = \{\emptyset, E, \{a\}, \{c, d\}, \{b, c, d\}\}$:
   We have $\{a\} \cup \{c, d\} = \{a, c, d\} \notin \tau_1$. Thus τ_2 is not a topology.
   iii) $\tau_3 = \{\emptyset, E, \{a\}, \{a, b\}, \{a, b, c\}\}$:
   $*)$ $\emptyset, E \in \tau_3$,
   $**)$ $\forall \mathcal{O} \in \tau_3 : \emptyset \cap \mathcal{O} = \emptyset \in \tau_3, E \cap \mathcal{O} = \mathcal{O} \in \tau_3$,
   $\{a\} \cap \{a, b\} = \{a\} \in \tau_3,\ \{a\} \cap \{a, b, c\} = \{a\} \in \tau_3, \{a, b\} \cap \{a, b, c\} = \{a, b\} \in \tau_3$
   $***)$ $\forall \mathcal{O} \in \tau_3 : \emptyset \cup \mathcal{O} = \mathcal{O} \in \tau_3, E \cup \mathcal{O} = E \in \tau_3, \{a\} \cup \{a, b\} = \{a, b\} \in \tau_3$,
   $\{a\} \cup \{a, b, c\} = \{a, b, c\} \in \tau_3, \{a, b\} \cup \{a, b, c\} = \{a, b, c\} \in \tau_3$,
   Hence, τ_3 is a topology.
2. The closed sets of the topology τ_1 are $\{\emptyset, E, \{b, c, d\}, \{a, b\}, \{a\}\}$.
   The closed sets of the topology τ_3 are $\{\emptyset, E, \{b, c, d\}, \{c, d\}, \{d\}\}$.

Exercice 02 : $a \in \mathbb{R}, I_\alpha =]\alpha, +\infty[$, $\tau = \{\emptyset, \mathbb{R}, I_\alpha(\alpha \in \mathbb{R})\}$.

1. Show that $(\mathbb{R}, \tau)$ is a topological space :
   $*)$ $\emptyset, \mathbb{R} \in \tau$,
   $**)$ $\forall \mathcal{O} \in \tau : \emptyset \cap \mathcal{O} = \emptyset \in \tau, \mathbb{R} \cap \mathcal{O} = \mathcal{O} \in \tau$,
   Let $\{I_{\alpha_i}\}_{1 \le i \le n} \subset \tau$. We put $\alpha = \max\limits_{1 \le i \le n} \alpha_i$. We have : $\bigcap\limits_{i=1}^{n} I_{\alpha_i} = I_\alpha \in \tau$.
   $***)$ $\forall \mathcal{O} \in \tau : \emptyset \cup \mathcal{O} = \mathcal{O} \in \tau, \mathbb{R} \cup \mathcal{O} = \mathbb{R} \in \tau$,
   Let $\{I_{\alpha_i}\}_{i \in I} \subset \tau$. We set $\alpha = \min\limits_{i \in I} \alpha_i$.
   If $\alpha = -\infty$ then we find $\bigcup\limits_{i \in I} I_{\alpha_i} = \mathbb{R} \in \tau$ If $\alpha > -\infty$ then we get $\bigcup\limits_{i=1}^{n} I_{\alpha_i} = I_\alpha \in \tau$
   Hence $(\mathbb{R}, \tau)$ is a topological space.
2. We have $\emptyset \in (\mathbb{R}, |.|), \mathbb{R} \in (\mathbb{R}, |.|)$, and for $I_\alpha \in \tau$ is an interval , then $I_\alpha \in (\mathbb{R}, |.|)$. We obtain $\tau \subset (\mathbb{R}, |.|)$. Therefore τ is coarser, weaker, or smaller than $\mathbb{R}$.

Exercice 03 : $E = \{a, b, c\}$, $\tau = \{\emptyset, E, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$.

1. Show thatτ is a topology :
   $*)$ $\emptyset, E \in \tau$,

$**)$ $\forall \mathcal{O} \in \tau : \emptyset \cap \mathcal{O} = \emptyset \in \tau, E \cap \mathcal{O} = \mathcal{O} \in \tau, \{a\} \cap \{b\} = \emptyset \in \tau, \{a\} \cap \{a,b\} = \{a\} \in \tau,$ $\{a\} \cap \{a,c\} = \{a\} \in \tau, \{b\} \cap \{a,b\} = \{b\} \in \tau, \{b\} \cap \{a,c\} = \emptyset \in \tau, \{a,b\} \cap \{a,c\} = \{a\} \in \tau$

$***)$ $\forall \mathcal{O} \in \tau : \emptyset \cup \mathcal{O} = \mathcal{O} \in \tau, E \cup \mathcal{O} = E \in \tau, \{a\} \cup \{b\} = \{a,b\} \in \tau, \{a\} \cup \{a,b\} = \{a,b\} \in \tau,$

$\{a\} \cup \{a,c\} = \{a,c\} \in \tau, \{b\} \cup \{a,b\} = \{a,b\} \in \tau, \{b\} \cup \{a,c\} = E \in \tau, \{a,b\} \cup \{a,c\} = E \in \tau$

Thus τ is a topology.

2. $C_E^{\{a\}} = \{b,c\} \notin \tau, C_E^{\{a,b\}} = \{c\} \notin \tau$. Hence $\{a\}, \{a,b\}$ are not closed sets.

Exercice 04 : $\tau = \{\emptyset, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathcal{C}_{\mathbb{R}}^{\mathbb{Q}}, \mathcal{C}_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{N}, \mathcal{C}_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{Z}, \mathbb{R}\}$, $D = \{3, \sqrt{3}\}$.

1. i) $\mathcal{V}(D) = \{V \subset \mathbb{R}; \exists \mathcal{O} \in \tau : D \subset \mathcal{O} \subset V\}$. The open sets that containing D are : $\mathcal{C}_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{N}, \mathcal{C}_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{Z}, \mathbb{R}$,
   but, we have : $\mathcal{C}_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{N} \subset \mathcal{C}_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{Z} \subset \mathbb{R}$. Hence, $\mathcal{V}(D) = \{V \subset \mathbb{R} : \mathcal{C}_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{N} \subset V\}$.
   ii) $D' = \{x \in D, \forall V \in \mathcal{V}(x); V \cap (D \setminus \{x\}) \neq \emptyset\}$.
   * If $x = 3$, one has $D \setminus \{x\} = \{\sqrt{3}\}$, and one has $\mathbb{N} \in \mathcal{V}(x)$ but, $\mathbb{N} \cap (D \setminus \{x\}) = \emptyset$. Thus, $3 \notin D'$.
   * If $x = \sqrt{3}$, we have $D \setminus \{x\} = \{3\}$, in addition we have $\mathbb{Q} \in \mathcal{V}(x)$ but, $\mathbb{Q} \cap (A \setminus \{x\}) = \emptyset$. Then $\sqrt{3} \notin D'$.
   * If $x \in \mathbb{Q} \setminus \{3\}$, one has $D \setminus \{x\} = D$, and $\forall v \in \mathcal{V}(x); \cap (D \setminus \{x\}) = \{3\}$. Then, $x \in D'$.
   * If $x \in \mathbb{R} \setminus (\mathbb{Q} \cup \{\sqrt{3}\})$, we get $D \setminus \{x\} = D$, and $\forall v \in \mathcal{V}(x); \cap (D \setminus \{x\}) = \{\sqrt{3}\}$. We find $x \in D'$.
   Moreover ; $D' = \mathbb{R} \setminus D$.
   iii) $\overline{D} = D \cup D' = \mathbb{R}$, and $\overset{0}{D} = \emptyset$, hence, $\mathcal{F}r(D) = \overline{D} \setminus \overset{0}{D} = \mathbb{R}$, and $\mathcal{E}xt(D) = \emptyset$.
2. $\overline{D} = \mathbb{R}$. Thus, D is dense in $\mathbb{R}$.
   Conclusion : D is countable and everywhere dense in $\mathbb{R}$. Hence ; $(\mathbb{R}, \tau)$ is separable.
3. $\tau_{\mathbb{Z}} = \{\emptyset, \mathbb{N}, \mathbb{Z}\}$ and $Trivial_{\mathbb{Z}} = \{\emptyset, \mathbb{Z}\}$. So, the topology $\tau_{\mathbb{Z}}$ is finer than $Trivial_{\mathbb{Z}}$.

Exercice 05 : Determine the interior and the closure in $(\mathbb{R}, |\cdot|)$:

$A = \{-1 + \frac{1}{n}, n \in \mathbb{N}^*\} : \overset{0}{A} = \emptyset$, $\overline{A} = A \cup \{-1\}$.

$B =]-1, 1[\cup \{2\} \cup [3, 4[: \overset{0}{B} =]-1, 1[\cup]3, 4[$, $\overline{B} = [-1, 1] \cup \{2\} \cup [3, 4]$.

$C = \{x \in \mathbb{R} : x^2 \leq 4\} \cap [1, 5[: C = [1, 2]$, $\overset{0}{C} =]1, 2[$, $\overline{C} = [1, 2]$.

$D = \mathbb{Q} \cap [-1, 1]$, $\overset{0}{D} = \emptyset$, $\overline{D} = [-1, 1]$.

Exercice 06 : $\triangle = \{(x; x) : x \in E\}$

$\Longrightarrow$ Suppose that $\triangle$ is a closed set of E^2, so $\Omega = C_{E^2}^{\triangle}$ is an open set of E^2. let $x, y \in E^2$ be such that $x \neq y$, then $(x, y) \in \Omega$ and Ω is a neighborhood of (x, y), $\exists V_x \in \mathbf{V}(x), \exists V_y \in \mathbf{V}(y) : \quad V_x \times V_y \subset \Omega$, hence $V_x \cap V_y = \emptyset$.

$\Longleftarrow$ Let $(x, y) \in \Omega$, we have $x \neq y$, then $\exists V_x \in \mathbf{V}(x), \exists V_y \in \mathbf{V}(y) : \quad V_x \cap V_y = \emptyset$. So $V_x \times V_y \subset \Omega$. i.e. $\Omega \in \mathbf{V}(x, y), \quad \forall (x, y) \in \Omega$, we deduce $\Omega = C_{E^2}^{\triangle}$ is an open of E^2.
i.e. $\triangle$ is closed.

Exercice 07 :

$$\forall f, g \in E : d(f,g) = |f(0) - g(0)| + \int_0^1 |f(t) - g(t)|dt$$

1. Show that d is a distance on E :
   - Positive : $d \geq 0$ (Obvious).
     Let $f, g \in E$.
     $$\begin{aligned} d(f,g) = 0 \quad &\Leftrightarrow \quad |f(0) - g(0)| = 0 \wedge \int_0^1 |f(t) - g(t)|dt = 0 \quad \Leftrightarrow \quad |f(t) - g(t)| = 0, \forall t \in [0,1]. \\ &\Leftrightarrow \quad f(t) = g(t), \forall t \in [0,1]. \quad \Leftrightarrow \quad f \equiv g \end{aligned}$$
   - Symmetry : Obvious.
   - Triangle inequality : Let $f, g, h \in \mathbb{R}$. Then;
     $$\begin{aligned} d(f,h) &= |f(0) - h(0)| + \int_0^1 |f(t) - h(t)|dt \\ &\leq |f(0) - g(0)| + \int_0^1 |f(t) - g(t)|dt + |g(0) - h(0)| + \int_0^1 |g(t) - h(t)|dt \\ &= d(f,g) + d(g,h) \end{aligned}$$

   So, d is a distance on E.
2. Elements of the unit ball in (E, d) : $x, x^2, \sqrt{x}, x^\alpha (\alpha \in \mathbb{Q}^+)$.

Exercice 08 :

$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, $f : \overline{\mathbb{R}} \to [-1, 1]$ such that :

$$f(x) = \begin{cases} -1 & : \quad x = -\infty \\ \dfrac{x}{1+|x|} & : \quad x \in \mathbb{R} \\ 1 & : \quad x = +\infty \end{cases}$$

*) Let us show that $d(x,y) = |f(x) - f(y)|$ defines a distance on $\overline{\mathbb{R}}$: We must show that f is bijective.

- Positive : $d \geq 0$ (Obvious).
  Soit $x, y \in \overline{\mathbb{R}}$.
  $$\begin{aligned} d(x,y) = 0 \quad &\Leftrightarrow \quad |f(x) - f(y)| = 0 \\ &\Leftrightarrow \quad f(x) = f(y) \\ &\Leftrightarrow \quad x = y \text{ (since } f \text{ est bijective)} \end{aligned}$$
- Symmetry : Obvious.
- Triangle inequality : Obvious

**) $B(0,1) = \{x \in \overline{\mathbb{R}} : d(0,x) < 1\} = \mathbb{R}$.

***) $\overline{B}(0,1) = \{x \in \overline{\mathbb{R}} : d(0,x) \leq 1\} = \overline{\mathbb{R}}$.

Exercice 09 :

$-U = \{-x : x \in U\}$ $\qquad$ $\lambda U = \{\lambda x : x \in U\}$ $(\lambda \in \mathbb{R}^*)$ $\qquad$ $a + U = \{a + x : x \in U\}$ $(a \in \mathbb{R})$

Montrer que :

1. $\Rightarrow$ Let $x \in -U$. Hence : $-x \in U$. Then, there exists $r > 0$ such that $B(-x, r) =]-x-r, -x+r[\subset U$.
   Thus; $]x-r, x+r[\subset -U$, i.e. $-U$ is open.
   $\Leftarrow$ $-U$ open $\Rightarrow U = -(-U)$ open.

2. $\Rightarrow$ Assume that $\lambda > 0$ and let $x \in \lambda U$. Thus, $\dfrac{x}{\lambda} \in U$. Then, there exists $r > 0$ such that $]\dfrac{x}{\lambda} - r, \dfrac{x}{\lambda} + r[\subset U$.
   Hence ; $]x - \lambda r, x + \lambda r[\subset U$, ie. λU is open.
   For all $\lambda < 0$: U open $\Rightarrow -\lambda U$ open $\Rightarrow \lambda U$ open.
   $\Leftarrow \lambda U$ open $\Rightarrow U = \dfrac{1}{\lambda}(\lambda U)$ open.
3. $\Rightarrow$ Let $x \in a+U$. Then, $x-a \in U$. Then, there exists $r > 0$ such that $]x-a-r, x-a+r[\subset U$.
   Hence, $]x - r, x + r[\subset a + U$, ie. $a + U$ is open.
   $\Leftarrow a + U$ open $\Rightarrow U = -a + (a + U)$ open.

Exercice 10 : Let $x, y, z \in E$. We have

1. $d(x, y) = 0 \Longleftrightarrow \|x - y\| = 0 \Longleftrightarrow x - y = 0 \Longleftrightarrow x = y$,
2. $d(x, y) = \|x - y\| = |-1|\|y - x\| = d(y, x)$,
3. $d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$,

Hence, d is a distance on E.

"Homework N01 is evaluated out of 5 points, from a total of 20."
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