University of Mohamed Boudiaf-M'sila **L2 :** *Semester 03*
Faculty of Mathematics and Computer Science **S01 :** *Topological Spaces*
Department of Mathematics **Year :** *2024-2025*
Module: Introduction to Topology

Exercice 01 : Let $E = \{a, b, c, d\}$ be a set.

1. Determine the topology in the following families : $\tau_1 = \{\emptyset, E, \{a\}, \{c, d\}, \{a, c, d\}\}$, $\tau_2 = \{\emptyset, E, \{a\}, \{c, d\}, \{b, c, d\}\}$, $\tau_3 = \{\emptyset, E, \{a\}, \{a, b\}, \{a, b, c\}\}$.
2. In topology cases, find the closed sets ?

Exercice 02 : Let $\alpha \in \mathbb{R}$, $I_\alpha =]\alpha; +\infty[$ and $\tau = \{\emptyset, \mathbb{R}, I_\alpha, (\alpha \in \mathbb{R})\}$.

1. Prove that $(\mathbb{R}, \tau)$ is a topological space ?
2. Compare between τ and the Euclidean topology (standard) of $\mathbb{R}$?

Exercice 03 : Let $E = \{a, b, c, d\}$ be a set endowed with the topology $\tau = \{\emptyset, E, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$.

1. Show that τ is a topology on E ?
2. Are $\{a\}, \{a, b\}$ closed sets ?

Exercice 04 : Let $\mathbb{R}$ be a real line endowed with the topology $\tau = \{\emptyset, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathcal{C}_{\mathbb{R}}^{\mathbb{Q}}, \mathcal{C}_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{N}, \mathcal{C}_{\mathbb{R}}^{\mathbb{Q}} \cup \mathbb{Z}, \mathbb{R}\}$ and assume that $D = \{3, \sqrt{3}\}$.

1. Find the family of neighbourhoods of D, $(\mathcal{V}(D))$, the accumulation point of D, (D'), the boundary of D, $(\mathcal{F}r(D))$, and the exterior of the set D, $(\mathcal{E}xt(D))$?
2. Prove that D is dense in $\mathbb{R}$. Conclude ?
3. Compare between the induced topology $\tau_{\mathbb{Z}}$ and the trivial topology of $\mathbb{Z}$?

Exercice 05 : Let $(\mathbb{R}, |\cdot|)$ be a topological space. Find the interiors and the closure of the following subsets :

$A = \{-1 + \frac{1}{n}, n \in \mathbb{N}^*\}$
$B =]-1, 1[\cup \{2\} \cup [3, 4[$
$C = \{x \in \mathbb{R} : x^2 \leq 4\} \cap [1, 5[$
$D = \mathbb{Q} \cap [-1, 1]$

Exercice 06 : Let (E, τ) be a topological space. We equip E^2 with the product topology. Show that E is Hausdorff (separated) if and only if $\triangle$ is closed in E^2.
where

$$\triangle = \{(x; x) : x \in E\}$$

(the diagonal of E^2).

Exercice 07 : Let $E = \mathcal{C}([0, 1], \mathbb{R})$. We define the function $d : E \times E \longrightarrow \mathbb{R}$ as follows :

$$\forall f, g \in E : d(f, g) = |f(0) - g(0)| + \int_0^1 |f(t) - g(t)| dt$$

1. Show that d is a distance on E ?
2. Find the open ball of (E, d) ?

Exercice 08 : Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and let $f : \overline{\mathbb{R}} \to [-1, 1]$ such that :

$$f(x) = \begin{cases} -1 & : \quad x = -\infty \\ \dfrac{x}{1+|x|} & : \quad x \in \mathbb{R} \\ 1 & : \quad x = +\infty \end{cases}$$

Show that $d(x, y) = |f(x) - f(y)|$ define a distance on $\overline{\mathbb{R}}$ and find $B(0, 1), \overline{B}(0, 1)$.

Exercice 09 : Let U be a subset nonempty of $\mathbb{R}$ endowed with the standard topology $(\mathbb{R}, |.|)$. We define :

$-U = \{-x : x \in U\}$

$\lambda U = \{\lambda x : x \in U\}$ $(\lambda \in \mathbb{R}^*)$

$a + U = \{a + x : x \in U\}$ $(a \in \mathbb{R})$

Prove that :

1. U open $\Leftrightarrow -U$ open.
2. U open $\Leftrightarrow \lambda U$ open.
3. U open $\Leftrightarrow a + U$ open.

Exercice 10 : Let E be a vector space on $\mathbb{K} = \mathbb{R}$ où $\mathbb{C}$. We say a norm on E a map

$$\begin{aligned} \|\cdot\| : E &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto \|x\| \end{aligned}$$

such that : $\begin{cases} \|x\| = 0 \Longleftrightarrow x = 0, \\ \forall \alpha \in \mathbb{K}, \forall x \in E : \|\alpha x\| = |\alpha|\|x\|, & \text{(Homogeneity)}, \\ \forall x, y \in E : \|x + y\| \leq \|x + y\|, & \text{(Triangle inequality)}. \end{cases}$

Let $d : E \times E \longrightarrow \mathbb{R}^+$ define by $d(x, y) = \|x - y\|, \forall x, y \in E$.
Show that d is a distance on E ?

Homework N01 :

1. Give an example of subsets. $A, B \subset \mathbb{R}$ such that

$$A \cap B = \emptyset, \quad \overline{A} \cap B \neq \emptyset, \quad A \cap \overline{B} \neq \emptyset.$$

2. Let A, B be subsets in a topological space.
   Prove $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

3. On the plane $\mathbb{R}^2$ consider the family τ consisting of the empty set, $\mathbb{R}^2$ and all open discs $\{x^2 + y^2 < r^2\}, \quad r > 0$. i.e.

$$\tau = \{\emptyset, \mathbb{R}^2, \{x^2 + y^2 < r^2\}, \quad r > 0\}$$

   Prove that τ defines a topology and determine the closure of the hyperbola $xy = 1$.