Exercice 01 : Let $I =]0,1[\subset \mathbb{R}$ be an open interval equipped with the Euclidean distance $|\cdot|$ and let the family $\{O_x\}_{x\in I}$ such that $O_x = \left]\frac{x}{2}, 2x\right[$.

1. Show that $\{O_x\}_{x\in I}$ covers I.
   i.e. is a family of subsets contained in I such that :
$$I = \cup\{O_x\}_{x\in I}$$
2. Prove that I is not compact.

Exercice 02 : [**Alexandroff's compactness**]: Let Ω be a locally compact space, $\omega \notin \Omega$ and $X = \Omega \cup \{\omega\}$. We say open of X either an open of Ω, or a subset of the form $\Gamma \cup \{\omega\}$, or $\Gamma \subset \Omega$ is the complement of a compact of Ω. Prove that :

1. We have defined a Topology on X.
2. This Topology induced on Ω the initial Topology of Ω.
3. X is Hausdorff (separated).
4. X is compact.

Application : $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ the completed real line.

Exercice 03 : Let $(\mathbb{R}, |\cdot|)$ be a topological space. Which of the following subsets are compact ?

1. $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$
2. $B = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 2x = 1\}$
3. $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 2y > 1\}$

Hint : Check the exercice 05 series N01.

Exercice 04 : Let X be a discrete topological space. Under what condition is the space X connected ?

Exercice 05 : Study the connectedness of $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$.

Exercice 06 :

1. Prove that $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected in $\mathbb{R}^2$.
2. Study the connectedness of $A = \{(x,y) \in \mathbb{R}^2 : x.y > 1\}$.

Homework N03 :

1. Consider the metric space $\mathbb{Q}$ of rational numbers with the Euclidean distance.
   Prove that
$$A = \{x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2}\}$$
   is closed and bounded, yet not compact.

2. Let $X \subset \mathbb{R}^2$ denote the union of the segment $\{x = 0,\ |y| \leq 1\}$ with the range of the function $f :]0,\ +\infty[\to \mathbb{R}^2$ given by

$$f(t) = (\frac{1}{t}, \cos(t))$$

Prove that X is closed in $\mathbb{R}^2$, connected and not path connected.

Hint : A topological space X is path(wise)-connected if, given any two points $x, y \in X$, there is a continuous mapping $\alpha : [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. Such an α is called a path from x to y.