Exercice 01 :
Let $X = \{a, b, c, d\}$ be a set equipped with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$, $Y = \{1, 2, 3, 4\}$ equipped with $\sigma = \{\emptyset, Y, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$, and suppose that

$$f : (X, \tau) \to (Y, \sigma)$$

is a map defined by : $f(a) = f(b) = 1, f(c) = 2, f(d) = 4$.

1. Calculate $\mathcal{V}(a), \mathcal{V}(b), \mathcal{V}(c), \mathcal{V}(d)$?
2. Calculate $\mathcal{V}(1), \mathcal{V}(2), \mathcal{V}(3), \mathcal{V}(4)$?
3. Calculate $f^{-1}(\mathcal{V}(1)), f^{-1}(\mathcal{V}(2)), f^{-1}(\mathcal{V}(4))$?
4. Study the continuity of f at a, b, c, d ?

Exercice 02 :
Let (X, τ) be a topological space and $A \subset X$. We define the indicator map of A (noted χ_A) from (X, τ) to $(\mathbb{R}, |.|)$ by

$$\chi_A(x) = \begin{cases} 1, & if \quad x \in A \\ 0, & if \quad x \notin A \end{cases}$$

Give a necessary and sufficient condition for the indicator application χ_A to be continuous.

Exercice 03 :
Let $E = C([0, 1]; \mathbb{R})$ is equipped with the following distances

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| dx \qquad d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

and assume that the map H from E to $(\mathbb{R}, |.|)$ defined by $H(f) = \int_0^1 |f(x)| dx$.

1. Prove that H is Lipschitz function from (E, d_1) to $(\mathbb{R}, |.|)$?
2. Prove that H is Lipschitz function from (E, d_∞) to $(\mathbb{R}, |.|)$?
3. Is H a bijective map ?

Exercice 04 :
Is the following sets closed in $(\mathbb{R}^2, d_2)$?

1. $A = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$
2. $B = \{(x, y) \in \mathbb{R}^2 : y \leq x^2\}$
3. $C = \{(x, y, z) \in \mathbb{R}^3 : z \leq x^2 - y^2 + 5\}$

Exercice 05 :
Build homeomorphisms between :

1. Two intervals of the form $[a; b]$ and $[c; d]$ with $a < b$ and $c < d$.

2. The interval $]-1;1[$ and $\mathbb{R}$.
3. The circle $C(0,1)$ and $\mathbb{R}$.
4. The sphere $S(0,1)$ and $\mathbb{R}^2$.

Exercice 06 :

Let $d : \mathbb{Q}^* \times \mathbb{Q}^* \longrightarrow \mathbb{R}^+$ be a map such that $d(p,q) = \begin{cases} 0 & : \ p = q \\ \dfrac{1}{|p|} + \dfrac{1}{|q|} & : \ p \neq q \end{cases}$

1. Show that d is a distance on $\mathbb{Q}^*$?
2. Are the two sequences $u_n = \dfrac{1}{n}$, $u_n = n$ Cauchy sequences ?
3. Show that $(\mathbb{Q}^*, d)$ is not complete ?

Exercice 07 :

Let $E = \mathbb{N}^*$ be a set. We put for all $n, m \in E$:

$$d(m,n) = \begin{cases} 0 & : \ m = n \\ 10 + \dfrac{1}{m} + \dfrac{1}{n} & : \ m \neq n \end{cases}$$

1. Prove that d is a distance on E ?
2. Show that (E,d) is not complete ?
3. Let $f : E \to E$ be a map with $f(n) = n + 1$. Prove that for all $n, m \in E (n \neq m)$ we have $d(f(n), f(m)) < d(n,m)$, but f is not contraction.

Homework N02 :

Part 01 : (Contraction Theorem) Let (X,d) be a metric space. A function $f : X \longrightarrow X$ is called a contraction (mapping) if there exists a real number $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha \, d(x,y) \text{ for every } x, y \in X.$$

1. Prove that if (X,d) is complete, non-empty, and $f : X \longrightarrow X$ a contraction, there exists a unique point $c \in X$ such that $f(c) = c$.
2. Show that for every $x \in X$ the sequence $(f_n(x))$ converges to c ?

Part 02 : 1. Find a map $f : \mathbb{R} \longrightarrow \mathbb{R}$ without fixed points and such that

$$|f(x) - f(y)| < |x - y| \text{ for every } x, y \in \mathbb{R}.$$

2. Let (X,d) be complete and $U \subset X$ closed. Every Cauchy sequence in U is Cauchy in X, so it converges to a limit $x \in \overline{U}$; therefore, (A,d) is complete provided U is closed.