Contents

1 Real numbers

1.1 Archimedean property, density and integer part property

Definition 1.1. Let $x \in \mathbb{R}$

1. The integer part of x denoted as $[x]$ is the unique integer satisfying

$$[x] \leq x < [x] + 1$$

or equivalently

$$x - 1 < [x] \leq x.$$

2. A set A is said to be dense in $\mathbb{R}$ if

$$\forall x, y \in \mathbb{R}, x < y, \ \exists z \in A : x < z < y.$$

☞ **Example 1.1.**
- $[0.5] = 0$ because $0 \leq 0.5 < 1$.
- $[-1.5] = -2$ because $-2 \leq -1.5 < -1$.
- If $x \in \mathbb{Z}$ then $[x] = x$ because $x \leq x < x + 1$.

Theorem 1.1 (Archimedean property). we have

$$\forall x \in \mathbb{R}_+^*, y \in \mathbb{R}, \exists n \in \mathbb{N} : nx \geq y.$$

Proof. Divide through by x. Then the Archimedean property says that for every real number $a = \frac{y}{x}$, we can find $n \in \mathbb{N}$ such that $n \geq a$. In other words, says that the set of natural numbers $\mathbb{N}$ is not bounded above. Suppose for contradiction that $\mathbb{N}$ is bounded above. Then due to the least upper bound axiom, there is $b = \sup \mathbb{N}$. Therefore number $b - 1$ cannot be an upper bound for $\mathbb{N}$ as it is strictly less than b (the least upper bound). Thus there exists an $m \in \mathbb{N}$ such that $m > b - 1$. it follows that $n := m + 1 > b$. This is contradiction since b being an upper bound. □

Theorem 1.2. The following properties are equivalent

1. **Archimedean property** $\forall x \in \mathbb{R}_+^*, y \in \mathbb{R}, \exists n \in \mathbb{N} : nx \geq y$.
2. **integer part property:** $\forall x \in \mathbb{R}, \exists n \in \mathbb{Z} : n \leq x < n + 1$
3. $\mathbb{Q}$ **is dense in** $\mathbb{R}$, that is $\forall x, y \in \mathbb{R}, x < y, \exists r \in \mathbb{Q} : x < r < y.$

Proof.
- 1) $\Longrightarrow$ 2) Let $x \in \mathbb{R}$ be given. We want to show that there exists an integer $n \in \mathbb{Z}$ such that $n \leq x < n + 1$. Consider the set

$$S = \{n \in \mathbb{Z} : n \leq x\}.$$

Due to the Archimedean property, the set S is non empty. Indeed. There is $n \in \mathbb{Z} : -n \geq -x$ then $n \leq x$ so $x \in S$. Since S is bounded above by x. By the well-ordering property of integers, there exists a greatest element in S denoted as n. Since n is the greatest integer less than x, we have $n \leq x < n+1$. Therefore, we have shown that for any real number x, there exists an integer n such that $n \leq x < n+1$.

- 2) $\Longrightarrow$ 3). Given $x, y \in \mathbb{R} : x < y$. Due to 2) there exists $q \in \mathbb{Z}^*$ such that

  $$q - 1 \leq \frac{1}{y-x} < q.$$

  Which implies that

  $$1 < q(y-x)$$

  Then

  $$qx + 1 < qy$$

  By 2), there exists $p \in \mathbb{Z}$ such that $p - 1 \leq qx < p$. Hence

  $$qx < p \leq qx + 1 < qy$$

  Consequently, dividing by q, it follows $x < \frac{p}{q} < y$.

- 3) $\Longrightarrow$ 1). Given $x \in \mathbb{R}_+^*$, $y \in \mathbb{R}$. If $x \geq y$ it is enough to take $n = 1$. If not then $0 < x < y$. from 3), there are $p, q \in \mathbb{N}^*$ such that $\frac{p}{q} \geq \frac{y}{x}$ and then $px \geq qy \geq y$, $(q \geq 1)$.

□

Corollary 1.3. the irrational set $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Given $x, y \in \mathbb{R}$ such that $x < y$. form the density of $\mathbb{Q}$, there are $r_1, r_2 \in \mathbb{Q}$ such that $x < r_1 < r_2 < y$. We know that $\sqrt{2}$ is irrational and greater than 1. Then taking $z = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1) \notin \mathbb{Q}$ we obtain $r_1 < z < r_2$. □

1.2 Bounded subset in $\mathbb{R}$

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Theorem 1.4 (Characterisation of the supremum and infimum). Let A be a bounded subset of $\mathbb{R}$. Then

$$\alpha := \inf A \iff \begin{cases} \forall x \in A : x \geq \alpha & (\alpha \text{ is a lower bound of A}) \\ \forall \varepsilon > 0, \exists x_0 \in A : x_0 < \alpha + \varepsilon & (\alpha \text{ is greater than any lower bound}) \end{cases}$$

$$\beta := \sup A \iff \begin{cases} \forall x \in A : x \leq \beta & (\alpha \text{ is an upper bound of A}) \\ \forall \varepsilon > 0, \exists x_0 \in A : \alpha - \varepsilon < x_0 & (\beta \text{ is less than any upper bound}) \end{cases}$$

Definition 1.2 (Maximum and minimum)**.** Let A be a subset of $\mathbb{R}$.

1. A maximum of A, denoted as $\max A$, is the greatest element of A. That is

$$\max \in A \quad \text{and } \forall x \in A : x \leq \max A$$

2. A minimum of A, denoted as $\min A$, is the least element of A. That is

$$\min \in A \quad \text{and } \forall x \in A : x \geq \min A$$

☞ **Remark 1.1.** Let A be a bounded subset.

- $\max A$ is an upper bound of A.
- If $\sup A \in A$, then $\max A = \sup A$.
- If $\max A$ exists then $\sup A = \max A$. Indeed, since $\max A$ is an upper bound of A, it suffices to show that

$$\forall \varepsilon > 0, \exists x_0 \in A : \max A - \varepsilon < x_0.$$

  Given any $\varepsilon > 0$, we can take $x_0 = \max A$. Then we have $\max A - \varepsilon < \max A = x_0$.
- If $\sup A \notin A$, then $\max A$ does exists, because if not, $\sup A = \max A \in A$.
- Analogously for $\inf A$ and $\min A$.

☞ **Example 1.2.** Find $\sup A$, $\inf A$, $\max A$, $\min A$ if they exist, for the following cases.

1. Let $A := \{1, 2, 3\}$. We observe that $\min A = 1$, $\max A = 3$, leading to $\inf A = 1$ and $\sup A = 3$.
2. For $A =]0, 1]$, using the interval definition, we note that 0 is a lower bound, and 1 is an upper bound of A. Since $1 \in A$, we conclude that $\sup A = \max A = 1$. We now prove that $\inf A = 0$. Given $\varepsilon > 0$ (we can assume ε is arbitrarily small), if we choose $x_0 := \frac{\varepsilon}{2} \in A$, we have $x_0 < 0 + \varepsilon$. This shows that $\inf A = 0$. As $0 \notin A$, $\min A$ doesn't exist.

3. Let $A := \left\{\frac{n}{n^2+1} \mid n \in \mathbb{N}\right\}$. We observe that for all $n \in \mathbb{N}$, $0 < \frac{n}{n^2+1} \leq \frac{1}{2}$ (using $ab \leq \frac{1}{2}(a^2+b^2)$). Thus, $\frac{1}{2}$ is an upper bound of A. Since $\frac{1}{2} = \frac{1}{1^2+1} \in A$, we deduce $\max A = \sup A = \frac{1}{2}$. Moreover, we can prove 0 is the infimum of A. For any $\varepsilon > 0$, we observe that

$$\frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n}, \quad \frac{1}{n} \leq \varepsilon \Longleftrightarrow n \geq \frac{1}{\varepsilon}.$$

Due to the Archimedean property, choose n such that $n \geq \frac{1}{\varepsilon}$ (e.g., $n = \left[\frac{1}{\varepsilon}\right] + 1$). This guarantees $\frac{n}{n^2+1} \leq \frac{1}{n} \leq \varepsilon$ Thus, 0 is indeed the infimum of A. As $0 \notin A$, $\min A$ doesn't exist.