Contents

1 Complex numbers

1.1 Definition of the Field of Complex Numbers

Theorem 1.1. The set of complex numbers $\mathbb{C}$ is an Abiliene field with the following two operations:

1. Addition: For complex numbers $z = x + ix'$ and $z' = x' + iy'$, the sum $z + z'$ is defined as
$$z + z' = (x + x') + i(y + y')$$
2. Multiplication: The product $z \cdot z'$ is defined as
$$z \cdot z' = (xx' - yy') + i(xy' + x'y)$$
and in particular $i^2 = -1$.

☞ **Remark 1.1.** The field $\mathbb{C}$ extends the real numbers $\mathbb{R}$ by introducing the imaginary unit i, which satisfies $\boxed{i^2 = -1}$.

Proposition 1.2. The set $\mathbb{C}$ has the following properties:

- Multiplicative Inverse: Every non-zero complex number $z = x+iy$ has a multiplicative inverse z^{-1}, given by
$$z^{-1} = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$
- Identity Elements: The additive identity is $x + iy = 0 \iff x = y = 0$, and$0 \times i = 0$.

The field of complex numbers $\mathbb{C}$ is a fundamental mathematical structure that plays a significant role in various mathematical, scientific, and engineering applications.

Definition 1.1 (Conjugate and Modulus). The complex conjugate of $z = x + iy$ is $\bar{z} = x - iy$. The modulus (magnitude) of z is $|z| = \sqrt{x^2 + y^2}$.

☞ **Remark 1.2 (Subtraction and Division).** Since $\mathbb{C}$ is a field, then it follows

$$\boxed{z - z' = (x - x') + i(y - y')}$$

$$\boxed{\frac{z}{z'} = \frac{x + iy}{x' + iy'} = \frac{(x + iy)(x' - iy')}{x'^2 + y'^2}}$$

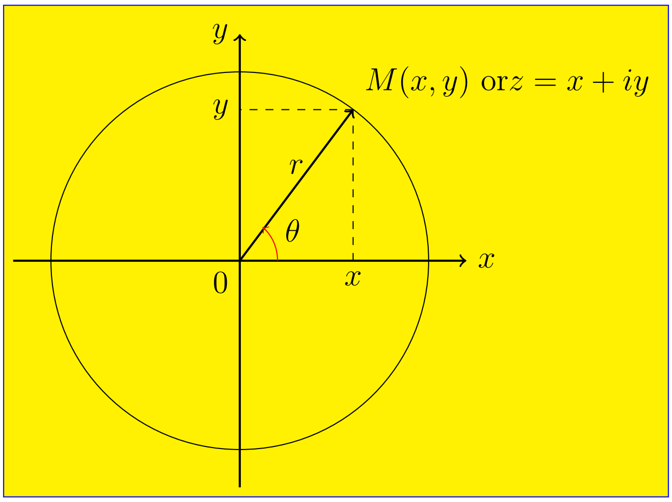
☞ **Remark 1.3.** Both the real part and the imaginary part of the complex number can represent a specific point in the plane. This representation aims to differentiate between the points representing real values and their complex counterparts during mathematical operations.

Proposition 1.3. We have the following useful properties.

- **Equality to Zero**:
$$|z| = 0 \iff z = 0$$
- **Triangle Inequality**:
$$|z + z'| \leq |z| + |z'|$$
- **Multiplicative Property**:
$$|zz'| = |z| \cdot |z'|$$
- **Modulus of the Conjugate**:
$$|z| = |\overline{z}|$$
- **Modulus of Quotient**:
$$\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|}, \text{ if } z' \neq 0$$
- **Modulus of a Complex Conjugate Product**:
$$z \cdot \overline{z} = |z|^2$$

1.2 Geometric Interpretation of Complex Numbers

In the complex number system, each complex number $z = x + iy$ can be associated with a point (x, y) in the complex plane. The complex plane is a two-dimensional plane where the horizontal axis represents the real part of the complex number (x), and the vertical axis represents the imaginary part (y).



The complex number $z = x + iy$ corresponds to the point (x, y) on the complex plane. The distance from the origin to this point is the modulus of the complex number, given by $r = \sqrt{x^2 + y^2}$. The angle θ that the line connecting the origin and the point makes with the positive real axis is the argument of the complex number.

Using this geometric interpretation, addition and subtraction of complex numbers correspond to vector addition and subtraction in the complex plane. Multiplication by a complex number corresponds to scaling and rotation, where multiplication by i results in a counterclockwise rotation by 90 degrees.

The geometric interpretation of complex numbers provides an intuitive way to understand their behavior and operations in terms of points and vectors on the complex plane.

1.3 Polar Form

Definition 1.2. A complex number $z = x + iy$ can also be represented in polar form as

$$z = r(\cos\theta + i\sin\theta),$$

where:

- r is the modulus (magnitude) of the complex number, given by $r = |z| = \sqrt{a^2 + b^2}$.
- θ is the argument (angle) of the complex number in the complex plane. It is characterized by :

$$\boxed{\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}, \ \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}.}$$

The following properties apply to the argument $\arg(z)$:

Proposition 1.4.

1. **Argument of a Real Positive Number:** For a positive real number x,

$$\arg(x) = 0$$

.

2. **Argument of a Non-Positive Real Number:** For a non-positive real number x,

$$\arg(x) = \pi$$

.

3. **Argument of a Pure Imaginary Number:** For a pure imaginary number yi, where y is a real number, $\arg(yi) = \frac{\pi}{2}$ if $y > 0$ and $\arg(yi) = -\frac{\pi}{2}$ if $y < 0$.

4. **Argument of a Product:** For two complex numbers z_1 and z_2,

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

.

5. **Argument of a Quotient:** For two complex numbers z_1 and z_2,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

.

6. **Argument of a Complex Conjugate:** For a complex number $z = a + bi$,

$$\arg(\overline{z}) = -\arg(z)$$

.

7. **Argument of the Reciprocal:** For a non-zero complex number z,

$$\arg\left(\frac{1}{z}\right) = -\arg(z)$$

.

8. **Argument of Powers:** For a non-zero complex number z and a positive integer n,

$$\arg(z^n) = n\arg(z)$$

.

1.4 Euler's Formula

Consider the function $f : \mathbb{R} \to \mathbb{C}$ defined by

$$f(\theta) = \cos\theta + i\sin\theta.$$

We observe that $\boxed{f(0) = 1, \text{ and its derivative is } f'(\theta) = if(\theta)}$. This observation prompts us to define $f(\theta) = e^{i\theta}$, and let us to the following definition.

Definition 1.3 (Euler's Formula). Euler's formula for complex numbers is as follows:

$$e^{i\theta} := \cos\theta + i\sin\theta.$$

Consequently, for any complex number z :

$$z = |z|e^{i\theta} = re^{i\theta}.$$

Exercise 1. Express the following complex numbers in algebraic form $(a + ib)$:

$$\boxed{\tfrac{1}{5+3i}, \ \tfrac{1}{(1+i)(1+i\sqrt{3})}}$$

Solution: We set $z = \frac{1}{(1+i)(1+i\sqrt{3})}$. First, we'll multiply both the numerator and denominator by the complex conjugate of the denominator to rationalize the expression:

$$
\begin{aligned}
z &= \frac{1}{(1+i)(1+i\sqrt{3})} \cdot \frac{(1-i)(1-i\sqrt{3})}{(1-i)(1-i\sqrt{3})} \\
&= \frac{(1-i)(1-i\sqrt{3})}{4 \times 2} = \frac{(1-i)(1-i\sqrt{3})}{8} = \frac{1+\sqrt{3}}{8} - i\frac{1+\sqrt{3}}{8}.
\end{aligned}
$$

Exercise 2. Calculate the cube roots of 1.

Solution: Let $z := re^{i\theta} \in \mathbb{C}$ be a cube root of 1. That is $z^3 = r^3 e^{3i\theta} = 1$. Hence

$$
\begin{cases} r^3 = 1 \\ 3\theta = 2n\pi \end{cases} \text{ and then } \begin{cases} r^3 = 1 \\ \theta = \frac{2n}{3}\pi \end{cases}
$$

Therefore

- if $n = 3k$, then $\theta = 2k\pi$. Hence $z = e^{2ik\pi} = 1$.
- if $n = 3k+1$, then $\theta = \frac{2\pi}{3} + 2k\pi$. Hence $z = e^{2i\pi/3 + 2ik\pi} = -1/2 + i\sqrt{3}/2$
- if $n = 3k+2$, then $\theta = \frac{4\pi}{3} + 2k\pi$. Hence $z = e^{4i\pi/3 + 2ik\pi} = -1/2 - i\sqrt{3}/2$.

Exercise 3.

1. Give the exponential form of the complex numbers: $1+i,\ 1+i\sqrt{3}$.
2. Calculate the real and imaginary parts of $\left(\frac{1+i\sqrt{3}}{1+i}\right)^{2022}$.

Solution:

1. Let $z_1 = 1+i,\ z_2 = 1+i\sqrt{3}$ and $\theta_1,\ \theta_2$ its argument respectively. We have $|z_1| = \sqrt{2}$ and $|z_2| = 2$. Therefore

$$
\begin{cases} \cos\theta_1 = 1/\sqrt{2} \\ \sin\theta_1 = 1/\sqrt{2} \end{cases} \text{ and } \begin{cases} \cos\theta_2 = 1/2 \\ \sin\theta_2 = \sqrt{3}/2 \end{cases}
$$

Then $\theta_1 = \pi/4 + 2n\pi$, $\theta_2 = \pi/3 + 2n\pi$. Hence

$$
z_1 = \sqrt{2}e^{i\pi/4}, \quad z_2 = 2e^{i\pi/3}.
$$

2. Let $z = \left(\frac{1+i\sqrt{3}}{1+i}\right)^{2022} = \left(\frac{z_2}{z_1}\right)^{2022}$. Then

$$
|z| = \left|\frac{z_2}{z_1}\right|^{2022} = \left(\sqrt{2}\right)^{2022} \text{ and } arg(z) = 2022\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{337\pi}{2} = \frac{\pi}{2} + 168\pi.
$$

Hence $z = (\sqrt{2})^{2022} i$.

Exercise 4. 1. Prove that $\forall z \in \mathbb{C} \setminus \{1\} : \frac{1+z}{1-z} \in i\mathbb{R} \iff |z| = 1$

2. Solve the equation: $z^3 = \overline{z}$

Solution:

1. Note that $z \in i\mathbb{R} \iff z = -\bar{z}$. Then

$$\begin{aligned}
\frac{1+z}{1-z} = i\mathbb{R} &\iff \frac{1+z}{1-z} = -\frac{1+\bar{z}}{1-\bar{z}} \\
&\iff (1+z)(1-\bar{z}) = -(1-z)(1+\bar{z}) \\
&\iff 1+z-\bar{z}-|z|^2 = -1+z-\bar{z}+|z|^2 \\
&\iff |z| = 1.
\end{aligned}$$

2. we set $z = re^{i\theta}$. Then $z^3 = r^3 e^{3i\theta}$ and $\bar{z} = re^{-i\theta}$. Hence

$$z^3 = \bar{z} \iff \begin{cases} r^3 = r \\ 3\theta = -\theta + 2n\pi \end{cases} \iff \begin{cases} (r=0) \vee (r=+1) \vee (r=-1 \text{ exclusive}) \\ (\theta = 2k\pi) \vee (\theta = \pi + 2k\pi) \end{cases}$$

Therefore there are three solutions :

- If $r = 0$ then $z = 0$.
- If $r = +1$ then $z = e^{i\pi} = 1$ or $z = e^{2in\pi} = -1$

Exercise 5. Let $\theta \in \mathbb{R}$. Calculate:

$$\boxed{A = \cos\theta + \cos(2\theta) + \cdots + \cos(n\theta)}$$

$$\boxed{B = \sin\theta + \sin(2\theta) + \cdots + \sin(n\theta)}$$

Solution: If $\theta = 2n\pi$, then $A = n$, $B = 0$. If not we have

$$\begin{aligned}
A + iB &= \sum_{k=}^{n} (e^{i\theta})^k = e^{i\theta}\frac{e^{in\theta}-1}{e^{i\theta}-1} = e^{i\theta}\frac{e^{in\theta/2}(e^{in\theta/2}-e^{-in\theta/2})}{e^{i\theta/2(}e^{i\theta/2}-e^{-i\theta/2})} \\
&= e^{i(n+1)\theta/2}\frac{2i\sin(n\theta/2)}{2i\sin(\theta/2)} = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\left(\cos((n+1)\theta/2) + i\sin((n+1)\theta/2)\right).
\end{aligned}$$

therefore

$$A = Re(A+iB) = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos((n+1)\theta/2) \text{ and } B = Im(A+iB) = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\sin((n+1)\theta/2)$$

Exercise 6. Simplify the following expression:

$$z = \frac{3+2i}{1-i}$$

Exercise 7. Solve the equation for z:

$$z^2 + 4z + 5 = 0$$

Exercise 8. Calculate the modulus and argument of the complex number $w = 2 + 2i$.

Exercise 9. Express $z = 3e^{i\pi/4}$ in the form $x + yi$, where x and y are real numbers.

Solution:

$$z = 3(\cos(\pi/4) + i\sin(\pi/4)) = 3/\sqrt{2} + 3i/\sqrt{2}.$$

Exercise 10. Given two complex numbers $u = -1 + 2i$ and $v = 3 - i$, calculate $u \cdot v$ and $\frac{u}{v}$.