University of Mohamed Boudiaf-Msila

Faculty of Sciences and Technologies **Academic Year 2023-2024.**

Module :Mathematics 01 **Renewable energy + License ST 1^{st} year**

Worksheet N°2

Exercise 01

Let $A = \{1;2;3\}$, $B = \{0;1;2;3\}$ and $E = \{0;1;2;3;4;5\}$

1. Find the sets:

- $A \cap B$.
- $A \cup B$.
- $A \setminus B$.
- $B \setminus A$.
- $A \Delta B$.
- $B \Delta A$.
- $C_E(A)$.
- $C_E(B)$..

2. Deducing card$(A \times B)$.

Exercise 02

a) Let $A =]-\infty;3]$, $B =]-2;7]$, and $C =]-5;+\infty[$ be three subsets of $\mathbb{R}$.

① Find:

1) $A \cap B$
2) $A \cup B$
3) $B \cap C$
4) $C_{\mathbb{R}}(A)$
5) $A \backslash B$
7) $A \Delta B$.

② Compare the following sets:

$$C_{\mathbb{R}}(A \cap B) \quad \text{and} \quad C_{\mathbb{R}}(A) \cup C_{\mathbb{R}}(B)$$

b) Let $F = \{2n; n \in \mathbb{N}\}$, Find $C_{\mathbb{N}}(F)$

Exercice 03 ⋆

Let E be a non-empty set, and A and B be two subsets of E. Show that:

① $A \subset B \iff C_E(B) \subset C_E(A)$

② $C_E(A \cap B) = C_E(A) \cup C_E(B)$

③ $C_E(A \cup B) = C_E(A) \cap C_E(B)$

Exercise 04 ⋆⋆

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = y$. What do the following formulas mean:

① $\forall x \in \mathbb{R}\ ,\ \exists! y \in \mathbb{R} : f(x) = y.$

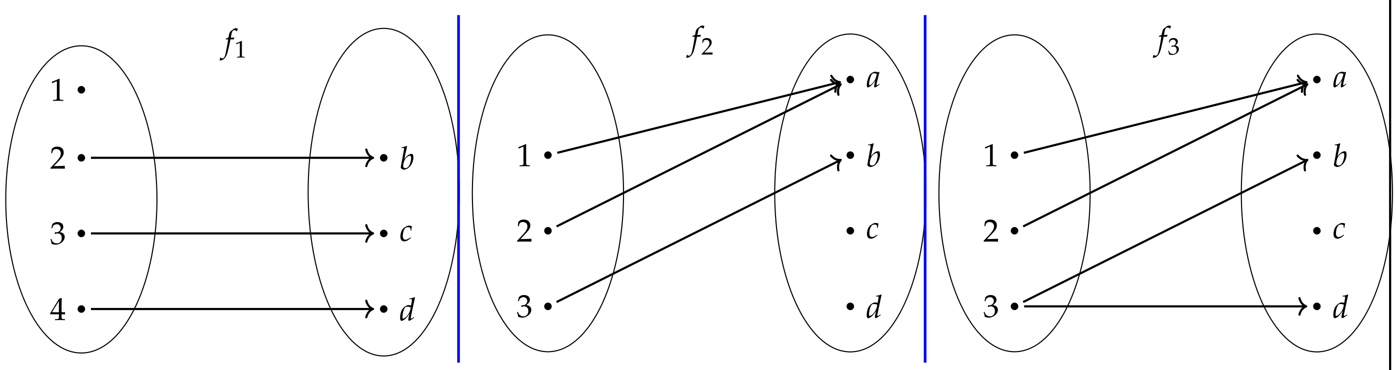
② $\forall y \in \mathbb{R}\ ,\ \exists x \in \mathbb{R} : f(x) = y$ (f is a map).

③ $(\forall x_1, x_2 \in \mathbb{R}$ si $f(x_1) = f(x_2)) \Rightarrow (x_1 = x_2)$. ($f$ is a map).

④ $\forall y \in \mathbb{R}\ ,\ \exists! x \in \mathbb{R} : f(x) = y.$ (f is a map).

Exercise 05 ⋆⋆

Which one among f_1, f_2, and f_3 is a function in the following cases?



Is f_2 injective? surjective?

Exercise 06 ⋆⋆ (direct image, inverse image".)

Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$f(x) = x^2 + 1$$

1. Find $f(2;6)$; $f(]2;6])$; and $f(]5;7])$, and then deduce $f(]2;6] \cup]5;7])$.

2. Determine $f^{-1}(\{2;6\})$; $f^{-1}(]2;6])$, $f^{-1}(]5;7])$,and then deduce $f^{-1}(]2;6]\cup]5;7])$ and $f^{-1}(]2;6]\cap]5;7])$.

3. Calculate $f(-2)$ and $f(2)$. Is f injective?

4. Determine $f(\mathbb{R})$. Is f surjective?

5. Determine a domain set I and a codomain set J for f to be bijective.

Remark

① $f(A_1\cup A_2)=f(A_1)\cup f(A_1)$.
② $f(A_1\cap A_2)\subset f(A_1)\cap f(A_1)$ and $f(A_1\cap A_2)=f(A_1)\cap f(A_1)$, if f is injective.
③ $f^{-1}(B_1\cap B_2)=f^{-1}(B_1)\cap f^{-1}(B_1)$.
④ $f^{-1}(B_1\cup B_2)=f^{-1}(B_1)\cup f^{-1}(B_1)$.

Exercise 07 ⋆⋆⋆ (Exam 2019-2020 (Univ-M'sila))

Let U be the function from $\mathbb{R}$ to $]-2,+\infty[$ defined by:

$$U(x)=e^x-2$$

1. Determine $U^{-1}(0)$ and $U((0,\ln(2)))$.

2. Show that U is bijective and determine U^{-1}.

Exercise 08 ⋆⋆ (Exam 2021-2022 (Univ-M'sila))

Let U be the function from $\mathbb{R}$ to the interval $]-1;+\infty[$ defined by

$$U(x)=e^{-2x+2}-1$$

1. Determine $U([-1;1])$ and $U^{-1}(0)$ **(2 points)**.

2. Show that U is bijective and determine U^{-1} **(3 points)**.

Exercise 09 ⋆⋆⋆ (Exam 2019-2020 (Univ-U.S.T.H.B))

Consider the function f defined from $\mathbb{R}$ to $\mathbb{R}$ by: $f(x) = e^{\cos(x)}$

1. Determine $f\left(\left[0; \frac{\pi}{2}\right]\right)$, $f\left(\left[-\frac{\pi}{2}; 0\right]\right)$, and $f^{-1}(\{3\})$.
2. Is f injective, surjective?

Exercise 10 ⋆: (Resit exam 2021 (Univ-M'sila))

Let $E = [1, 2]$ and $F = [1, 3]$, two intervals in $\mathbb{R}$. Consider the function $U : E \to F$, defined by

$$U(x) = \frac{x+2}{-x+4}$$

1. Show that U is increasing.
2. Determine $U([1; 1.5])$.
3. Show that for all $x \in [1, 2]$, $U(x) = -1 + \frac{6}{-x+4}$, then determine $U^{-1}\left(\left[1; \frac{5}{4}\right]\right)$.
4. Show that U is injective.
5. Show that if $x \in [1; 2]$, then $U(x) \in [1; 2]$.
6. Is U surjective?

Exercise 12 ⋆ (Exam 2018-2019 (Univ-U.S.T.H.B))

Consider the function f defined from $\mathbb{R}$ to $\mathbb{R}$ by: $f(x) = \frac{1}{1 + \ln(e + x^2)}$

1. Calculate the derivative of f and create its table of variations.
2. Determine the direct image $f(-1, 1)$ and the inverse image $f^{-1}(0)$.
3. Conclude that f is neither injective nor surjective.
4. Determine a domain set and a codomain set such that f is bijective. Find the expression for f^{-1} in this case.

Exercise 13

Check the injectivity and surjectivity of the following functions:

(1) $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x^2$

(2) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$

(3) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$

(4) $f : \mathbb{R} \to \mathbb{R}_+$ given by $f(x) = x^2$

(5) $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(x) = x^2$

(6) $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f(x) = x^2$

Exercise 14

Let $\mathbb{N}$ be the set of natural numbers, and the function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = 2n + 3$ for all $n \in \mathbb{N}$. Then f is

(A) surjective

(B) injective

(C) bijective

(D) none of these

Exercise 15

Let $\mathbb{Q}$ be the set of ratinal numbers, and the function $f : \mathbb{Q} \to \mathbb{Q}$ be defined by $f(n) = 2n + 3$ for all $n \in \mathbb{Q}$. Then f is

(A) surjective

(B) injective

(C) bijective

(D) none of these

Exercise 16

Which of the following functions are one-to-one, onto, or bijective? Justify your answer.

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$.
2. $g : \mathbb{N} \to \mathbb{N}$, $g(n) = n + 1$.
3. $h : \mathbb{N} \to \mathbb{N}^{\star}$, $h(n) = n + 1$.
4. $r : \mathbb{R} \to \{0\}$, $r(x) = 0$.

Exercise 17

Consider two functions $f : \left[0; \frac{\pi}{2}\right] \to \mathbb{R}$ given by $f(x) = \sin x$ and $g : \left[0; \frac{\pi}{2}\right] \to \mathbb{R}$ given by $g(x) = \cos x$.

1) Show that f and g are one-to-one (injective)
2) Is $h = f + g$ one-to-one?

Exercise 18

"Let f be the function defined from $\mathbb{R}$ to $\mathbb{R}$ by: $f(x) = \dfrac{1 - x^2}{x^2 + 1}$.

1. Solve $f(x) = \dfrac{1}{4}$. What can be inferred from this?
2. Solve $f(x) = 2$. What can be inferred from this?
3. Determine a condition on the real number a for the equation $f(x) = a$ to have solutions in $\mathbb{R}$.
4. Deduce an interval J in $\mathbb{R}$ such that $g : \mathbb{R} \to J$, $x \mapsto f(x)$, is surjective.
5. Find an interval I in $\mathbb{R}$ such that $h : I \to J$, $x \mapsto f(x)$, is bijective."

Exercise 19

Show that the function $f : \mathbb{R} \to]-1, 1[$ defined by

$$f(x) = \frac{x}{|x| + 1},$$

is one-to-one and onto function.

Exercise 20 ⋆⋆⋆

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by: $f(x) = \dfrac{1}{1+x^2}$, $\forall x \in \mathbb{R}$. And Let $A = \{-1, 1\}$; $B = [0, 1[$; and $C = [-1, 0]$.

1. Determine $f(A)$; $f^{-1}(A)$;$f(B)$; $f^{-1}(B)$; $f^{-1}(C)$.
2. Show that f is neither injective nor surjective.
3. Provide a domain set for f to be injective and a codomain set for f to be surjective.
4. Provide the expression for f^{-1}.

Exercise 21 ⋆⋆⋆

Let f and g be defined as follows: $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $f(x) = 2x$ and $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$g(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ \dfrac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$$

• Examine injectivity and surjectivity, then bijectivity of f and g. Determine $f \circ g$ and $g \circ f$.

.

Exercise 22 ⋆⋆⋆

Let E, F, and G be three sets, and let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be two functions.

1. Show that if $g \circ f$ is injective, then f is injective.
2. Show that if both f and g are injective, then $g \circ f$ is injective.
3. Show that if both f and g are surjective, then $g \circ f$ is surjective.
4. Deduce that if both f and g are bijective, then $g \circ f$ is bijective.

Exercise 23 ⋆⋆⋆

Consider the function f from $\mathbb{R}-$ to $\mathbb{R}+$ defined by: $f(x) = |x|^2$.

Is the function f bijective? If yes, find the function f^{-1}.

Exercise 24 ⋆⋆

Consider the function h from $\mathbb{R}-\{-1,+1\}$ to $\mathbb{R}-\{1\}$ defined by: $h(x)=\dfrac{2+|x|}{|x|-1}$.

1– For $a\in\mathbb{R}-\{-1,+1\}$, calculate $h(a)$ and $h(-a)$.

2– Is the function h injective?

Exercise 25 ⋆⋆⋆

Consider the function g from $\mathbb{R}/2$ to $\mathbb{R}/2$ defined by:

$$g(x)=\frac{1+2x}{x-2}.$$

1– Is the function g bijective? If yes, find the function g^{-1}.

2– Determine the function $g\circ g$.

Exercise 26 ⋆: (Exam SM 2018-2019 (Univ-M'sila))

Consider the function f from $\mathbb{R}/-1,+1$ to $\mathbb{R}/1$ defined by:

$$f(x)=\frac{3+x^2}{x^2-1}.$$

1– Is the function f surjective?

2– Is the function f injective?

3– Is the function f bijective?

Exercise 27

Let T a binary relation on $\mathbb{R}^\star$ defined as: $xTy \iff x\times y>0$.

1. Show that T is an equivalence relation.
2. Find the equivalence classes of 2.

Exercise 28

"Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^3-3x+2$, and let S be a relation in $\mathbb{R}$ defined by

$$xSy \iff f(x)=f(y)$$

1. Show that S is an equivalence relation.

.

2. Give the equivalence class of 2.

Exercise 29 ⋆ (Exam 2022 (Univ-A.M.BEDJAIA))

"Let $\mathcal{R}$ be the relation defined on $\mathbb{Z}$ as follows:

$$\forall x, y \in \mathbb{Z}, x\mathcal{R}y \iff \exists k \in \mathbb{Z}, x + y = 2k$$

.

(a) Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

(b) Determine the equivalence class of 0."

Exercise 30

Let $(L_i)_i$ be the set of all lines in a plane, and let $\mathcal{R}$ be the relation in $(L_i)_i$ defined as

$$L_i \mathcal{R} L_j \iff L_i \text{ is perpendicular to } L_j.$$

Show that $\mathcal{R}$ is symmetric but neither reflexive nor transitive.

Exercise 31

Let us define a relation on $\mathbb{N}$ by

$$a\mathcal{R}b \iff a \text{ divides } b.$$

show that $\mathcal{R}$ is a partial order .

Exercise 32

if the following relations are reflexive, symmetric, antisymmetric, and transitive:

1. $x, y \in \mathbb{Z}$ $\quad x\mathcal{R}y \Leftrightarrow x = -y.$
2. $x, y \in \mathbb{R}$ $\quad x\mathcal{R}y \Leftrightarrow \cos(x)^2 + \sin(y)^2 = 1.$

Deduce, among the mentioned relations, which one is an equivalence relation.

Exercise 33

Show that the relation $\mathcal{R}$ defined on $\mathbb{R}$ as $x\mathcal{R}y \Leftrightarrow |x| = |y|$ for $x, y \in \mathbb{R}$ is an equivalence relation.

Then, for x in $\mathbb{R}$, find the equivalence class of x .

Exercise 34

Show that the relation $\mathcal{R}$ defined on $\mathbb{R}$ as $x\mathcal{R}y \Leftrightarrow xe^y = ye^x$ for $x, y \in \mathbb{R}$ is an equivalence relation.

Exercise 35 ⋆⋆ :

We define on $\mathbb{R}$ the relation

$$\forall (x,y) \in \mathbb{R} : x\mathcal{R}y \iff x^2 - y^2 = x - y$$

.

1. Show that $\mathcal{R}$ is an equivalence relation.
2. Calculate the equivalence class of an element x. How many elements are there in this class?

Exercise 36 ⋆ : (Exam 2015-2016 (Univ-M'sila))

Let $\lambda \in \mathbb{R}$ and $\mathcal{R}$ is a binary relation defined by

$$\forall x, y \in \mathbb{R} : x\mathcal{R}y \iff x^3 - y^3 = \lambda(x^2 - y^2)$$

.

1. Show that $\mathcal{R}$ is an equivalence relation.
2. Let $\lambda = 7$, determine the equivalence class of 6.

Exercise 37 : ⋆⋆⋆ (Examen 2015-2016 (Univ-M'sila))

Let E be a set. We define on $P(E)$ (the set of subsets of E), the following relation:

$$A\mathcal{R}B \iff (A = B \; ou \; A = C_E(B)), \quad A, B \subset E$$

where $C_B(E)$ is the complement of B in E.

1. Prove that $\mathcal{R}$ is an equivalence relation.
2. For $E = \mathbb{R}$, determine the equivalence class of $A = [0;1]$, what do you notice?

Exercise 38 ⋆

Let $\prec$ be a binary relation on $\mathbb{R}^2$ defined as $(x,y) \prec (x_0,y_0) \Leftrightarrow x \leq x_0$ and $y \leq y_0$.

1. Demonstrate that $\prec$ is a partial order relation. (Is it a total order?)
2. Determine $\sup A$, $\inf A$, $\max A$, and $\min A$ for $A = \{(1,2),(3,1)\}$.
3. Is the set A linearly ordered?

Exercise 39 ⋆

Determine whether they exist: $\sup(A)$, $\max(A)$, $\min(A)$, and $\inf(A)$ in the following cases:

1. $A = \{-3;5\}$.
2. $A = [-1,1[$.
3. $A = \left\{\cos\left(\frac{2n\pi}{5}\right), n \in \mathbb{Z}\right\}$.
4. $A = \{2^n, n \in \mathbb{N}\}$.
5. $A = [-3;4] \cup [7;10]$.
6. $A = \left\{-1 + \frac{1}{2n+1}, n \in \mathbb{N}\right\}$.
7. $A = \left\{1 + \frac{1}{2n}, n \in \mathbb{N}^*\right\}$.
8. $A = \left\{(-1)^n + \frac{1}{n}, n \in \mathbb{N}^*\right\}$.
9. $A = \{n^2 - 4n + 3, n \in \mathbb{N}\}$.

Exercise 40 ⋆⋆ (Exam 2018-2019 (Univ-U.S.T.H.B))

Let $A = \left\{3 - \dfrac{1}{2n+1}, n \in \mathbb{N}\right\}$. Determine whether they exist: $\sup(A)$, $\max(A)$, $\min(A)$, and $\inf(A)$.