Contents

1 Sequences

1.1 Subsequence

Definition 1.1. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence and $k_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers. Then the sequence $(u_{k_n})_{n\in\mathbb{N}}$ is called a subsequence of $(u_n)_{n\in\mathbb{N}}$.

☞ **Example 1.1.**
- The sequences $(u_{2n})_{n\in\mathbb{N}}$, $(u_{2n+1})_{n\in\mathbb{N}}$ are sub-sequences of $(u_n)_{n\in\mathbb{N}}$ (with $k_n = 2n,\ k_n = 2n+1$ respectively).
- The sequence $(u_{6n})_{n\in\mathbb{N}}$ is a subsequence of $(u_n)_{n\in\mathbb{N}}$, with $k_n = 6n$ and it is a subsequence of $(u_{2n})_{n\in\mathbb{N}}$ with $k_n = 3n$.

Proposition 1.1. If the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent, then every subsequence $(u_{k_n})_{n\in\mathbb{N}}$ is also convergent and we have $\lim_{n\to+\infty} u_{k_n} = \lim_{n\to+\infty} u_n$.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence with limit ℓ, and let $(u_{k_n})_{n\in\mathbb{N}}$ be a subsequence (indexed by natural numbers k_n, where $k_0 < k_1 < k_2 < k_3 < \dots$). Since $(u_n)_{n\in\mathbb{N}}$ converges to ℓ, for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N : |u_n - \ell| \leq \varepsilon$$

Now, since $(u_{k_n})_{n\in\mathbb{N}}$ is a subsequence, then $k_n \to +\infty$ as $n \to +\infty$. so we can find N' such that

$$\forall n \geq N' : k_n \geq N.$$

By the convergence of $(u_n)_{n\in\mathbb{N}}$, we have

$$n \geq N' \Longrightarrow k_n \geq N \Longrightarrow |u_{k_n} - \ell| \leq \varepsilon$$

This satisfies the definition of convergence of the sub sequence. □

Theorem 1.2. Every bounded sequence $(u_n)_{n\in\mathbb{N}}$ has convergent subsequence.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence. This means that there exists a constant $M > 0$ such that $|u_n| \leq M$ for all $n \in \mathbb{N}$.

Consider the closed interval $[u_1 - M, u_1 + M]$. Since the sequence is bounded, all of its terms must lie within this interval. Now, divide this interval into two closed subintervals of equal length: $[u_1 - M, u_1]$ and $[u_1, u_1 + M]$.

At least one of these subintervals must contain infinitely many terms of the sequence $(u_n)_{n\in\mathbb{N}}$. Let's denote the chosen subinterval as I_1.

Next, divide I_1 into two equal subintervals and proceed similarly: choose the one that contains infinitely many terms of the sequence. Denote this subinterval as I_2.

Continue this process recursively. At the k-th step, divide the current interval into two equal subintervals and choose the one containing infinitely many terms of the sequence. Denote this subinterval as I_k.

We now have a nested sequence of closed intervals:

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

By the nested interval property of real numbers, there exists a unique point c that belongs to all of these intervals:

$$c \in \bigcap_{k=1}^{\infty} I_k$$

Since each interval I_k contains infinitely many terms of the sequence, it follows that c is a limit point of the sequence. Therefore, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converging to c.

Thus, every bounded sequence has a convergent subsequence. □