Chapter 6

Tutorial Sessions

6.1 Tutorial 1

Exercise 1 Let $\left(L, I^0, (\varphi_\alpha)_{\alpha\in I^0}, (\psi_\alpha)_{\alpha\in I^1}, n, N\right)$ be a Lukasiewicz multivalent algebra with involution L:

Show that the following conditions are equivalent for an element x of L:

(i) $x \in C(L)$;

(ii) $\exists y \in L, \exists i \in I^0$ such that $x = \varphi_i(y)$;

(iii) $\exists i \in I^0$ such that $x = \varphi_i(x)$;

(iv) $\forall i \in I^0, x = \varphi_i(x)$;

(iv) $\forall i, j \in I^0, \varphi_i(x) = \varphi_j(x)$.

Exercise 2 Show that any involutive multivalent Lukasiewicz algebra is a Kleene algebra, i.e.,

$$x \wedge Nx \leq y \vee Ny, (\forall x, y \in L)$$

Exercise 3 Let $\wp(E)$ be the set of fuzzy subsets of a finite set $E = \{x, y\}$ with $J = \left\{0, \frac{1}{2}, 1\right\}$;

1. Provide $\widetilde{\wp(E)}$.

2. Draw the Hasse diagram of $(\widetilde{\wp(E)}, \subset)$.

3. Show that $\left(\widetilde{\wp(E)}, \subset, J^0, (N_\alpha)_{\alpha\in J^0}, (N'_\alpha)_{\alpha\in J^1}, n, C\right)$ is an involutive trivalent Lukasiewicz algebra.

4. Let $\left(L, I^0, (\varphi_\alpha)_{\alpha\in I^0}, (\psi_\alpha)_{\alpha\in I^1}, n, N\right)$ be an involutive multivalent Lukasiewicz algebra. Show that it can be embedded in an algebra of fuzzy subsets.

6.2 Tutorial 2

Exercise

A- We know that an algebra $(L, \to, N, 1)$ of type $(2, 1, 0)$ is equivalent to a Lukasiewicz trivalent algebra $(L, \wedge, \vee, N, 0, 1, \mu)$ via the transformations

$(\mathbf{LW}) x \to y = (\mu Nx \vee y) \wedge (\mu Ny \vee x) = [(\mu Nx \wedge y) \vee Nx \vee y]$ and

(WL1) $x \vee y = (x \to y) \to y$,

(WL2) $x \wedge y = N(Nx \vee Ny)$,

(WL3) $\mu x = Nx \to x$,

(WL4) $N0 = 1$,

if and only if

(W1) $x \to (y \to x) = 1$,

(W2) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,

(W3) $((x \to Nx) \to x) \to x = 1$,

(W4) $(Nx \to Ny) \to (y \to x) = 1$,

(W5) $1 \to x = 1 \Rightarrow x = 1$,

(W6) $x \to y = 1$ and $y \to x = 1 \Rightarrow x = y$.

1. Show that the relation defined on L by $x \leq y$ if and only if $x \to y = 1$ is a partial order on L.

2. Show that if $x \leq y$, then $y \to z \leq x \to z$.

3. Show that $(x \to Nx) \to x = x$, and $Nx \leq x \to y$.

4. Show that $NNx = x$, and $Ny \to Nx = x \to y$

5. Show that $x \leq y \Rightarrow Ny \leq Nx$ and $1 \to x = x$

6. Show that $x \leq x \vee y$ and $x \leq y \Leftrightarrow x \vee y = y$.

7. Show that $x \to (y \to z) = y \to (x \to z)$.

B- Let $\left(L, I^0, (\varphi_\alpha)_{\alpha \in I^0}, (\psi_\alpha)_{\alpha \in I^1}, n, N\right)$ be an involutive multivalent Lukasiewicz algebra.

1. Show that any involutive multivalent Lukasiewicz algebra is a Kleene algebra, i.e., $x \wedge Nx \leq y \vee Ny, (\forall x, y \in L)$

2. Now, suppose that $L = \{1, 2, \ldots, p-1\}$.

Show that $N(i) = p - i$, for all $i \in L$.

6.3 Tutorial 3

Exercise 1

Definition: Let G be a group. A fuzzy subset A of the group G is called a fuzzy subgroup of G if:

i. $\mu_A(xy) = \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in G$;

ii. $\mu_A(x^{-1}) = \mu_A(x)$ for all $x \in G$.

Definition: Let G be a group, e denote the identity element of the group G. A fuzzy subset A of the group G is called a fuzzy subgroup of G if:

i. $\mu_A(xy^{-1}) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in G$;

ii. $\mu_A(e) = 1$.

1. Show that a fuzzy subset A of the group G is a fuzzy subgroup of G if and only if: $\mu_A(xy^{-1}) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in G$.

2. Let A be a fuzzy subgroup of the group G and x an element of G then:

$\mu_A(xy) = \mu_A(y)$ for all $y \in G$ if and only if $\mu_A(x) = \mu_A(e)$.

Exercise 2

Let $(G, .)$ be a group, i.e., a set equipped with a binary operation denoted by dot, which is associative, has an identity element denoted by 1, and such that for every x in G, there exists an inverse x' satisfying $x.x' = x'.x = 1$. A subgroup is a subset of G stable under the inverse and the binary operation. Show that if A is a fuzzy subset of G, then the following are equivalent:

$\forall x \in G \mu A(x') \geq \mu A(x)$

$\Leftrightarrow \forall x, y \in G, \mu_A(x.y') \geq \min(\mu_A(x), \mu_A(y))$

$\Leftrightarrow \forall \alpha \in [0, 1], A_\alpha$ is a subgroup of G.

Then we say that A is a fuzzy group in G.

6.4 Tutorial 4

Exercise 1

Let $X = [0, 1]$ with $\alpha, \beta \in R$ and let $a, b \in R$. Define the fuzzy set A on X as follows:

$$\mu A(x) = \begin{cases} 0, \text{ if } x < a - \alpha \text{ or } b + \beta < x \\ 1, \text{ if } a < x < b \\ 1 + x - \alpha a, \text{ if } a - \alpha < x < a \\ 1 - b - \beta x, \text{ if } b < x < b + \beta \end{cases}$$

Determine $\text{Ker}(A), \text{Supp}(A)$ and $H(A)$.

Exercise 2

1. Determine their union and intersection.

2. Give the complement of A_1

3. Draw the diagrams of the union, intersection, and complement of A_1.

Exercise 3

Let $X = \{1, 2, 3, \ldots, 10\}$ and A a fuzzy subset of X given by:

$$A = \{< 1, 0.2 >, < 2, 0.5 >, < 3, 0.8 >, < 4, 1.0 >, < 5, 0.7 >, < 6, 0.3 >, <$$

$$7, 0.0 >, < 8, 0.0 >, < 9, 0.0 >, < 10, 0.0 >\}$$

Determine all α-cuts of A.

Exercise 4

1. $T_0(x, y) = \begin{cases} 0, (x, y) \in [0, 1[^2 \\ \min(x, y) \text{ otherwise.} \end{cases}$

2. $T_1(x, y) = \max(x + y - 1, 0)$.

3. $T_{1.5}(x, y) = 2 - x - xyy + xy$.

4. $T_2(x, y) = xy$.

5. $T_{2.5}(x, y) = x + xyy - xy$.

6. $T_3(x, y) = \min(x, y)$.

Show that we have: $T_0 \leq T_1 \leq T_{1.5} \leq T_2 \leq T_{2.5} \leq T_3$.

Exercise 5

Let the fuzzy sets A, B, and C defined on real numbers by the

membership functions $\mu_A(x) = \frac{x}{x+1}, \mu_B = \frac{1}{x^2+10}, \mu_C = \frac{1}{10^X}$. Determine the membership functions for:

a) $A \cup B, A \cap B$,

b) $A \cup B \cup C, A \cap B \cap C$,

c) $A \cap \overline{C, B} \cup C$,

d) $\overline{A \cap B}, \bar{A} \cup \bar{B}$

Exercise 6

Show that the two fuzzy sets satisfy the De Morgan's law.

$\mu_A(x) = \frac{1}{1+(x-10)}$,

$\mu_B = \frac{1}{1+x^2}$

6.5 Tutorial 5

Exercise 1 Let X be a non-empty set, and $R : X^2 \to [0,1]$ be a fuzzy relation.

Show that R is an order relation if and only if R_α is a crisp order relation, $\alpha \in]0,1]$.

Exercise 2 An involutive Lukasiewicz trivalent algebra

is an algebra $(L, \wedge, \vee, N, 0, 1, \mu)$ of type $(2, 2, 1, 0, 0, 1)$ such that

0. $(L, \wedge, \vee, N, 0, 1)$ is a De Morgan algebra,

1. $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$,

2. $\mu(x \vee y) = \mu(x) \vee \mu(y)$,

3. $\mu(x) \wedge N\mu(x) = 0$,

4. $\mu(\mu(x)) = \mu(x)$,

5. $\mu(N\mu(x)) = N\mu(x)$,

6. $N\mu Nx \leq \mu(x)$,

7. $\mu(x) = \mu(x) \& \mu(Nx) = \mu(Nx) \Rightarrow x = y$,

8. $\mu(x) \wedge Nx = x \wedge Nx$,

9. $\mu(x) \vee Nx = 1$,

10. $N\mu Nx \vee (\mu(x) \wedge \mu(Nx)) \vee \mu Nx = 1$,

11. $x \leq \mu x$,

12. $x \wedge Nx \leq x \vee Nx$.

a. Show that (0) & (8) & (9) $\Rightarrow$ (11).

b. Show that the following axiom systems are equivalent:

$$\mathbf{S0} = \{(\mathbf{0}), \ldots, (\mathbf{7})\}$$

$$\mathbf{S1} = \{(\mathbf{0}), (\mathbf{1}), (\mathbf{4}), (\mathbf{5}), (\mathbf{8}), (\mathbf{10}), (\mathbf{11})\}$$

$$\mathbf{S2} = \{(\mathbf{0}), (\mathbf{1}), (\mathbf{8}), (\mathbf{9})\}$$

$$\mathbf{S3} = \{(\mathbf{0}), (\mathbf{8}), (\mathbf{9}), (\mathbf{12})\}.$$

6.6 Final Exam, Algebraic Logic, February 2021

Exercise 1 (10 pts) Let $\left(L, I, (\varphi_\alpha)_{\alpha\in I^0}, (\psi_\alpha)_{\alpha\in I^1}, n, N\right)$ be a multivalent Lukasiewicz algebra with involution.

1. Give the formula for N if $|I| = 3$.

2. Could we find this formula if $|I| \geq 4$ (cardinality of $I \geq 4$)?

3. Show that if the chain I is finite, then the involution N is unique.

4. We are in the framework of a $\mathcal{L}_{3-}$ algebra $(L, \wedge, \vee, 1, 0, N, \mu)$ (in the sense of Moisil's first definition). Show the equivalences.

(a) $Nx \vee \mu x = 1$.

(b) $x \vee \gamma x = 1$.

(c) $\eta x \vee \mu x = 1$.

Exercise 2 (6 pts) Let X be a non-empty set, and $R : X^2 \to [0,1]$ be a fuzzy relation.

Show that R is an order relation if and only if R_α is a crisp order relation for all $\alpha \in]0,1]$.

Exercise 3 (4 pts) Let A, B be fuzzy sets defined on $\mathbb{R}$ by membership functions $\mu_A(x) = \frac{1}{(x-1)^2+1}$ and $\mu_B(x) = \frac{1}{x^2+1}$.

Determine the membership functions of each of the following fuzzy sets: $A \cup B, A \cap B, \overline{A \cap B}, \bar{A} \cup \bar{B}$.

6.6.1 Final Exam Correction

Exercise 1 (10 pts)

Let $\left(L, I, (\varphi_\alpha)_{\alpha\in I^0}, (\psi_\alpha)_{\alpha\in I^1}, n, N\right)$ be a multivalent Lukasiewicz algebra with involution.

1. If $|I| = 3$, we are in the case of a trivalent Lukasiewicz algebra, and the involution N is given by the formula: $Nx = \eta x \vee (x \wedge \gamma x)$. $\to (1.50pts)$

2. If $|I| \geq 4$, this question is still open. $\rightarrow (1.00pts)$

3. If the chain I is finite, then the involution N is unique. Indeed, assuming that $|I| = p$ and we have two decreasing involutions N_1 and N_2, then:

$$\begin{array}{lllllll} \varphi_1 N_1 & = & N_1\varphi_{p-1} & = & \overline{\varphi_{p-1}} & = & \varphi_1 N_2 \\ \varphi_2 N_1 & = & N_2\varphi_{p-2} & = & \overline{\varphi_{p-2}} & = & \varphi_2 N_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{p-1} N_1 & = & N_1\varphi_1 & = & \overline{\varphi_1} & = & \varphi_1 N_2 \end{array} \longrightarrow (1.50pts)$$

Thus, by Moisil's determination principle, we obtain $N_1 = N_2$.

4. Proof of equivalences:

$(a) \Rightarrow (b)$ Assume that $Nx \vee \mu x = 1$. We assume that: $Nx \vee \mu x = 1$, i.e., (a). Replace x with Nx, we get: $Nx \vee \mu x = NNx \vee \mu Nx = 1$. This implies $x \vee \gamma x = 1$. So, $(a) \Rightarrow (b)$. $\rightarrow (1.50pts)$

$(b) \Rightarrow (a)$ Assume that: $x \vee \gamma x = 1$. By replacing x with Nx in (b), we obtain $Nx \vee \gamma Nx = 1 \Rightarrow Nx \vee \mu x = 1$. So, $(a) \Leftrightarrow (b)$. $\rightarrow (1.5pts)$

$(b) \Rightarrow (c)$ Assume that $x \vee \gamma x = 1$. By replacing x with μx in (b) $x \vee \gamma x = 1$. Then $\mu x \vee \gamma\mu x = 1 (\gamma\mu = \mu N\mu = \mu\bar{\mu} = \bar{\mu} = N\mu = \eta)$. Thus, $\mu x \vee \eta x = 1$. So, $(b) \Rightarrow (c)$. Finally. $\rightarrow (1.50pts)$

$(c) \Rightarrow (b)$ It suffices to replace x with Nx and use the fact that $\mu Nx = \gamma x$ and $x \geq \vartheta x$. $\rightarrow (1.50pts)$

Exercise 2 (06 pts)

Let's assume that $R : X \times X \rightarrow [0, 1]$ is an order relation.

$*$ $R(x, x) = 1$, for all $x \in X$, then $(x, x) \in R_\alpha$, for all $\alpha \in]0, 1]$, i.e., R_α is reflexive, for all $\alpha \in]0, 1]$. $\rightarrow (1pt)$

$**$ Assume that $(x, y) \in R_\alpha$, and $(y, x) \in R_\alpha$. Then $R(x, y) \wedge R(y, x) \geq \alpha \Rightarrow x = y$, i.e., R_α is antisymmetric. $\rightarrow (1pt)$

$***$ Assume that $(x, y) \in R_\alpha, (x, z) \in R_\alpha$. Then $R(x, z) \geq R(x, y) \wedge R(x, z) \geq \alpha$, $(x, z) \in R_\alpha$, so R_α is transitive. $\rightarrow (1pt)$

Conversely, if R_α is an order relation $\forall\alpha \in]0,1]$. Let's show that R is an order relation.

$*$ R_1 is an order relation, so $R(x,x) \geq 1$, thus $R(x,x) = 1$. $\rightarrow (1pt)$

$**$ Let $x \neq y$ with $R(x,y) \wedge R(y,x) = \alpha$. Then $(x,y) \in R_\alpha$ and $(y,x) \in R_\alpha$, and by antisymmetry of R_α, we get $\alpha = 0$. $\rightarrow (1pt)$

$***$ Let $x, y, z \in X$. Set $R(x,y) \wedge R(x,z) = \lambda$.

Since R_λ is transitive, $(x,z) \in R_\lambda$. So $R(x,y) \wedge R(x,z) \leq R(x,z)$. $\rightarrow (1pt)$

In conclusion, R is an order relation $\Leftrightarrow$ R_α is an order relation $\forall\alpha \in]0,1]$.

Exercise 3 (04 pts)

A is defined on $\mathbb{R}\backslash\{1\}$ and B is defined on $\mathbb{R}\backslash\{0\}$

We study the sign of $d(x) = \mu_A x - \mu_B(x)$.

$d(x) = \mu_A(x) - \mu_B(x) = \frac{1}{(x-1)^2+1} - \frac{1}{x^2+1} = \frac{2x-1}{((x-1)^2+1)(x^2+1)}$

We have $\left((x-1)^2+1\right)\left(x^2+1\right) > 0$.

So $d(x)$ has the same sign as $2x-1$, or $d(x) > 0$, if $x \geq \frac{1}{2}$.

Alternatively, $\mu_A(x) \geq \mu_B(x)$ if $x \geq \frac{1}{2}$. $\rightarrow (0.5pt)$

In conclusion:

$$\mu_{A\cup B}(x) = \max\left(\mu_A(x), \mu_B(x)\right) = \begin{cases} \mu_A(x), \text{ si } x \geq \frac{1}{2}; \\ \mu_B(x), \text{ si } x < \frac{1}{2}. \end{cases} \rightarrow (1pt)$$

$$\mu_{A\cap B}(x) = \min\left(\mu_A(x), \mu_B(x)\right) = \begin{cases} \mu_B(x), \text{ si } x \geq \frac{1}{2} \\ \mu_A(x), \text{ sinon.} \end{cases} \rightarrow (1pt)$$

$$\mu_{\overline{A\cap B}}(x) = 1 - \min\left(\mu_A(x), \mu_B(x)\right) = \begin{cases} 1 - \mu_B(x), \text{ si } x \geq \frac{1}{2}; \\ 1 - \mu_A(x), x < \frac{1}{2}. \end{cases} \rightarrow (0,5pt)$$

$$\mu_{\bar{A}\cup\bar{B}}(x) = \max\left(\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)\right) = \max\left(1 - \mu_A(x), 1 - \mu_B(x)\right) \rightarrow (0,5pt)$$

$$= \begin{cases} 1 - \mu_B(x), \text{ si } x \geq \frac{1}{2} \\ 1 - \mu_A(x), \text{ si } x < \frac{1}{2} \end{cases} \rightarrow (0,5pt)$$

$$= \mu_{\overline{A\cap B}}(x)$$