Fuzzy Sets

Proposed by Ladeh Zadeh in 1965

"Fuzzy sets," Information and Control, vol. 8, pp. 338--353, 1965.

A generalization of set theory that allows partial membership in a set.

In the fuzzy theory, fuzzy set A of universe X is defined by function $\mu_A(x)$ called the *membership function* of set A

$\mu_A(x): X \rightarrow [0, 1]$, where $\mu_A(x) = 1$ if x is totally in A;
$\mu_A(x) = 0$ if x is not in A;
$0 < \mu_A(x) < 1$ if x is partly in A.

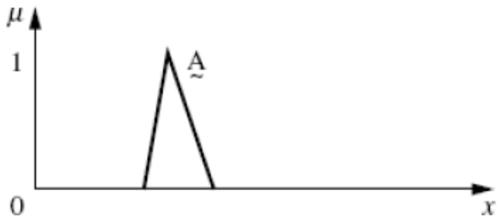


FIGURE 2.14
Membership function for fuzzy set $\underset{\sim}{A}$.

$\mu_A(x): X \rightarrow [0, 1]$

Definition 1. *Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function*

$$\mu_A : X \rightarrow [0, 1]$$

and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

It is clear that A is completely determined by the set of tuples

$$A = \{(x, \mu_A(x)) | x \in X\}$$

Frequently we will write simply $A(x)$ instead of $\mu_A(x)$. The family of all fuzzy (sub)sets in X is denoted by $\mathcal{F}(X)$. Fuzzy subsets of the real line are called *fuzzy quantities*.

If $X = \{x_1, \ldots, x_n\}$ is a finite set and A is a fuzzy set in X then we often use the notation

$$A = \mu_1/x_1 + \ldots + \mu_n/x_n$$

where the term $\mu_i/x_i,\ i = 1,\ldots,n$ signifies that μ_i is the grade of membership of x_i in A and the plus sign represents the union.

Discrete Form of a Fuzzy Set

$$\underset{\sim}{A} = \left\{ \frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1} + \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2} + \cdots \right\} = \left\{ \sum_i \frac{\mu_{\underset{\sim}{A}}(x_i)}{x_i} \right\}$$

Continuous Form of a Fuzzy Set

$$\underset{\sim}{A} = \left\{ \int \frac{\mu_{\underset{\sim}{A}}(x)}{x} \right\}$$

Example 1. *Suppose we want to define the set of natural numbers "clo to 1". This can be expressed by*

$$A = 0.0/-2 + 0.3/-1 + 0.6/0 + 1.0/1 + 0.6/2 + 0.3/3 + 0.0/4.$$

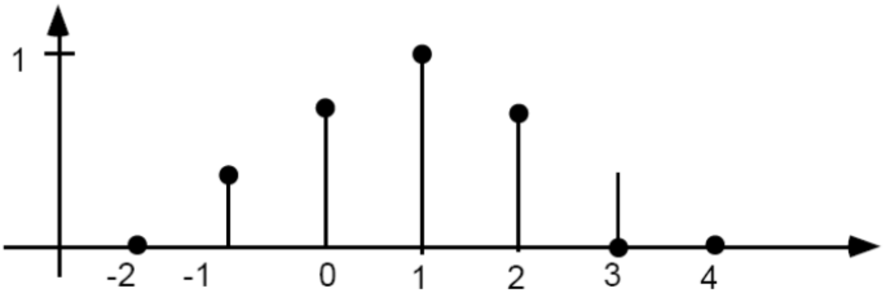


Figure 1. A discrete membership function for "x is close to 1".

Example 2 *The membership function of the fuzzy set of* real *numbers "close to 1", is can be defined as*

$$A(t) = \exp(-\beta(t-1)^2)$$

where β is a positive real number.

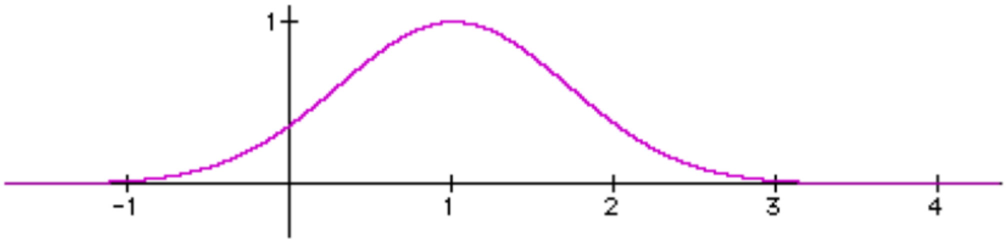


Figure 2 A membership function for "x is close to 1".

Fuzzy sets operations

- **Inclusion:** Let $A, B \in F(X)$. We say that the set A is included in B if

$$A(x) \leq B(x), \forall x \in X$$

The empty (fuzzy) set $\emptyset$ is defined as $\emptyset(x) = 0, \forall x \in X$, and the total set x is $X(x) = 1, \forall x \in X$.

Intersection: Let $A, B \in F(X)$. The intersection of A and B is the fuzzy set C with

$$C(x) = \min\{A(x), B(x)\} = A(x) \wedge B(x), \forall x \in X$$

Union: Let $A, B \in F(X)$. The union of A and B is the fuzzy set D with

$$D(x) = \max\{A(x), B(x)\} = A(x) \vee B(x), \forall x \in X$$

Complementation: Let $A \in F(X)$ be a fuzzy set. The complement of A is the fuzzy set B given by

$B(x) = 1 - A(x), \forall x \in X.$

We denote $B = \bar{A}$.

Remark The two principles of classical logic (the non contradiction and the excluded teirs) no longer remains valid in the theory of fuzzy sets i.e. $A \cap \bar{A} \neq \emptyset, A \cup \bar{A} \neq X$.

Fuzzy partition

Let A be a crisp set in universal set X and $\bar{A}$ be a complement set of A. The conditions $A \neq \emptyset$ and $A \neq X$ result in couple the $(A, \bar{A})$ which decomposes X into 2 subsets.

Definition
(Fuzzy partition) In the same manner, consider a fuzzy set satisfying $A \neq \emptyset$

and $A \neq X$. The pair $(A, \bar{A})$ is defined as fuzzy partition. Usually, if m subsets are defined in X, m-tuple $(A_1, A_2, A_3, \dots, A_n)$ holding the following conditions is called a fuzzy partition.

(i) $\forall i, \quad A_i \neq \emptyset$

(ii) $A_i \cap A_j = \emptyset$ for $i \neq j$,

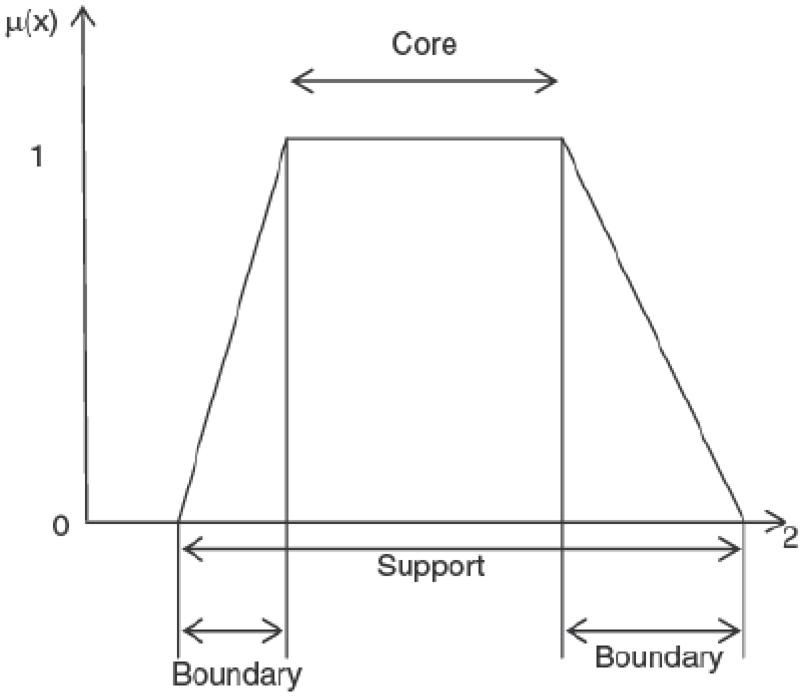
(iii) $\forall x \in X, \quad \sum_{i=0}^{m} \mu_{A_i}(x) = 1.$

Characteristics of fuzzy subsets

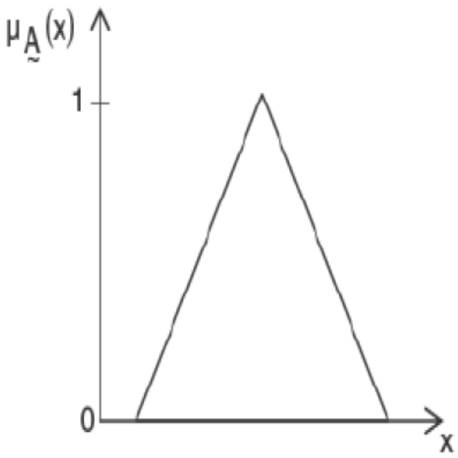
Definition 2 *(support) Let A be a fuzzy subset of X; the* support *of A, denoted $supp(A)$, is the crisp subset of X whose elements all have nonzero membership grades in A.*

$$supp(A) = \{x \in X | A(x) > 0\}.$$

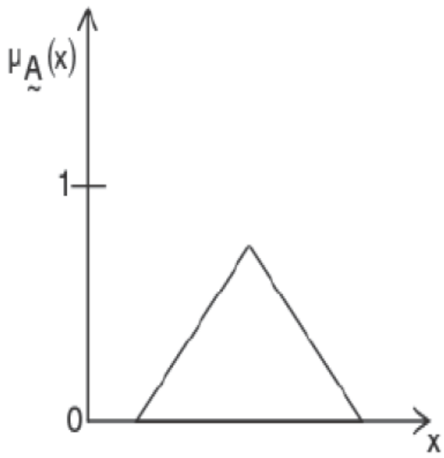
Definition 3 *(normal fuzzy set) A fuzzy subset A of a classical set X is called* normal *if there exists an $x \in X$ such that $A(x) = 1$. Otherwise A is subnormal.*



The kernel of A is the crisp subset on X given by
$\mathrm{Ker}(A) = \{x \in X / \mu_A(x) = 1\}$



normal



sub normal

(Height of fuzzy subset) Let A be a fuzzy set on a set X. The height of A is the highest value taken by its membership function

$H(A) = \sup\{\mu_A(x)/x \in X\}$

(Cardinality of a fuzzy subset)The cardinality of a finite fuzzy subset A denoted $|A|$ is defined by

$$|A| = \sum_{x \in X} \mu_A(x)$$

Example Let $X = \{1, 2, \ldots, 6\}$, and A be a fuzzy set of X given by:

$$A = \{\langle x, \mu_A(x)\rangle\} = \{\langle 1, 0.2\rangle, \langle 2, 0.0\rangle, \langle 3, 0.8\rangle, \langle 4, 1.0\rangle, \langle 5, 0.5\rangle, \langle 6, 1.0\rangle\}.$$

Then $\mathrm{supp}(A) = \{1, 3, 4, 5, 6\}, \mathrm{Ker}(A) = \{4, 6\}, H(A) = \{1\}, |A| = 3.5.$

Proposition

Let A a fuzzy subset of X. The kernel and support of a fuzzy subset verify the following properties:

$\mathrm{supp}\,(A^c) = (k\,\mathrm{er}(A))^c$

$\ker\,(A^c) = (\mathrm{supp}(A))^c$

Other fuzzy subset operations

$$A \oplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$$

$$\mu_{A \oplus B}(x) = \mathrm{Max}\,\{\mathrm{Min}\,[\mu_A(x), 1 - \mu_B(x)]\,, \mathrm{Min}\,[1 - \mu_A(x), \mu_B(x)]\}$$

Example

$$A=\{(x_1,0.2),(x_2,0.7),(x_3,1),(x_4,0)\}$$

$$B=\{(x_1,0.5),(x_2,0.3),(x_3,1),(x_4,0.1)\}$$

consequence,

$$A\oplus B=(A\cap\bar{B})\cup(\bar{A}\cap B)=\{(x_1,0.5),(x_2,0.7),(x_3,0),(x_4,0.1)\}$$

(Disjoint sum)

$$\mu_{A\Delta B}(x)=|\mu_A(x)-\mu_B(x)|$$

(Bounded difference) For novice-operator θ, we define the membership

$$\mu_{A\theta B}(x)=\text{Max}\,[0,\mu_A(x)-\mu_B(x)]$$

Distance in Fuzzy Set

(1) Hamming distance

$$d(A,B)=\sum_{i=0}^{n}|\mu_A(x_i)-\mu_B(x_i)|$$

Example Following A and B for instance,

$$A=\{(x_1,0.4),(x_2,0.8),(x_3,1),(x_4;0)\}$$

$$B=\{(x_1,0.4),(x_2,0.3),(x_3,0),(x_4;0)\}$$

$$d(A,B)=|0|+|0.5|+|1|+|0|=1.5$$

(2) Euclidean distance

$$e(A,B)=\sqrt{\sum_{i=0}^{n}(\mu_A(x_i)-\mu_B(x_i))^2}$$

Example Euclidean distance between sets A and B used for the previous Hamming distance is

$$e(A,B)=\sqrt{0^2+0.5^2+1^2+0^2}$$

Cartesian product and Projection of fuzzy subsets

Definition

(Cartesian product) The Cartesian product applied to n fuzzy sets can be

defined as follows: Let $\mu_{A_1}(x), \mu_{A_2}(x), \mu_{A_3}(x), \ldots, \mu_{A_n}(x)$ as membership function of $A_1, A_2, A_3, \ldots, A_n$. Then, the membership degree of $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$ on the fuzzy sets $A_1 \times \ldots \times A_n$ is,

$$\mu_{A_1 \times A_2 \times \ldots \times A_n} = \min\left[\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\right]$$

Example Lets $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2,\}$ and lets A_1, A_2 are two fuzzy subsets respectively defined on X and Y given by : $A_1 = \{\langle x_1, 0.1\rangle ; \langle x_2, 0.4\rangle ; \langle x_3, 0.75\rangle\}$, and $A_2 = \{\langle y_1, 0.2\rangle ; \langle y_2, 0.6\rangle\}$. So, we find:

$\mu_{A_1 \times A_2} = \{\langle (x_1, y_1), 0.1\rangle ; \langle (x_1, y_2), 0.1\rangle ; \langle (x_2, y_1), 0.2\rangle ; \langle (x_2, y_2), 0.4\rangle ;$

$\langle (x_3, y_1), 0.2\rangle ; \langle (x_3, y_2), 0.6\rangle\}$