M'sila University
Fcaulty of Mathematics and Computer
Department of Mathematics
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Algebra 4 course

TD Number 1

Exercise 1.

Find the matrix associated to each one of the following bilinear forms with respect to the canonical basis:

a. $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \quad f((x, y), (x', y')) = xx' + 3xy' - yx' + 2yy'$

b. $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \quad f((x_1, y_1), (y_1, y_2)) = x_1y_2 - x_2y_1$

c. $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \quad f((x, y, z), (x', y', z')) = xx' + 2xz' - 4yy' + 2zx' + 8zz'$

Exercise 2.

Indicate whether the bilinear forms in **Exercise 1.** are symmetric, asymmetric or neither one thing nor the other.

Exercise 3.

a. Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a bilinear form whose associated matrix relative to the canonical basis is $F_C = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$. Obtain $f((1, 1), (3, 2))$.

b. Let $f : M_{2\times2}(\mathbb{R}) \times M_{2\times2}(\mathbb{R}) \longrightarrow \mathbb{R}$ be a bilinear form whose associated matrix relative to the canonical basis is $F_C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 3 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$.

Obtain $f\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\right)$.

Exercise 4.

Let B be the canonical basis of $\mathbb{R}^2$ and let $B' = \{(3,2),(1,1)\}$.

a. $F_B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is the matrix associated to a certain bilinear form relative to the basis B. Obtain its associated matrix $F_{B'}$ relative to the basis B'.

b. $F_{B'} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ is the matrix associated to a certain bilinear relative to the basis B'. Obtain its associated matrix F_B relative to the basis B.

Exercise 5.

Let V be the space of real 2×2 matrices. Consider the bilinear form $\langle A, B \rangle = trace(AB)$.

a. Compute the matrix of the form with respect to the standard basis $B = \{e_1, e_2\}$.

b. Calculate the signature of this form. Is it a positive definite form?

c. Find an orthogonal basis for V.

d. Let W be the subspace of V of trace zero matrices. Determine the signature of the form restricted to W.

Exercise 6.

For the bilinear form $\varphi(p,q) = \int_0^1 p(x)q(x)dx$ on $P_2(\mathbb{R})$, find $[\varphi]_B$ for the basis $B = \{1, x, x^2\}$.