# Chapter 1

# Bilinear and Quadratic forms

## 1.1 Bilinear forms

**Definition 1.**

Bilinear form of a vector space $V$ is a function $\varphi$ of two variables on $V$, with values in the field $F$ satisfying the bilinear axioms which are:

$$\varphi(u_1 + u_2, v) = \varphi(u_1, v) + \varphi(u_2, v)$$
$$\varphi(u, v_1 + v_2) = \varphi(u, v_1) + \varphi(u, v_2)$$
$$\varphi(\alpha u, v) = \alpha\varphi(u, v) = \varphi(u, \alpha v)$$

for all $u, v \in V$ and $\alpha \in F$
Bilinear form will be denoted by $\langle u, v\rangle$

**Remark.** $\langle 0, v\rangle = \langle v, 0\rangle = 0$

### Symmetric bilienar forms

**Definition 2.**

A bilinear form is said to be symmetric if

$$\langle u, v\rangle = \langle v, u\rangle$$

and skew symmetric if

$$\langle u, v\rangle = -\langle v, u\rangle$$

**Examples.**

- $\varphi(x, y) = \langle x, y\rangle$ in $\mathbb{R}^n$ is bilinear and symmetric for any scaler product.
- $\varphi((x_1, y_1), (x_2, y_2)) = x_1y_1+2x_1y_2+3y_1x_2+4y_1y_2$ is bilinear, but not symmetric.

### Proprieties of a bilinear form

Let $V$ be a vector space and let $\varphi$ be a bilinear form on $V$. We have

$$\begin{gathered}\varphi(u+v, u+v) = \varphi(u, u) + 2\varphi(u, v) + \varphi(v, v)\\ \varphi(u-v, u-v) = \varphi(u, u) - 2\varphi(u, v) + \varphi(v, v)\\ \varphi(u, v) = \tfrac{1}{2}(\varphi(u+v, u+v) - \varphi(u, u) - \varphi(v, v))\\ \varphi(u, v) = \tfrac{1}{4}(\varphi(u+v, u+v) - \varphi(u-v, u-v))\end{gathered}$$

**Definition 3.**

A $n \times n$ matrix $A$ is called symmetric if $A^t = A$

**Theorem 1.**

Bilinear form given in above example is symmetric if and only if matrix $A$ is symmetric.

*Proof.* Assume that $A$ is symmetric. Since $V^tAU$ is a $1 \times 1$ matrix, it is equal to its transpose: $V^tAU = (V^tAU)^t = U^tA^tV = U^tAV$ and hence $\langle V, U\rangle = \langle U, V\rangle$ and it follows that form is symmetric. Conversely, let the form is symmetric. Set $U = e_i$ and $V = e_j$ where $e_i$ and $e_j$ are elements of fixed basis. We find that $\langle e_i, e_j\rangle = e_i^tAe_j = a_{ij}$ while $\langle e_j, e_i\rangle = e_j^tAe_i = a_{ji}$ and as the form is symmetric we get that $a_{ij} = a_{ji}$ and the matrix $A$ is symmetric. □

### Computation of the value of bilinear form

Let $u, v \in V$ and let $U$ and $V$ be their coordinates in the basis $B$ so that $u = BU$ and $v = BV$. Then

$$\langle u, v\rangle = \langle \sum_i u_ix_i, \sum_j v_jy_j\rangle$$

This expands using bilinearity to $\sum_{i,j} x_iy_j\langle u_i, v_j\rangle = \sum_{i.j} x_ia_{ij}y_j = U^tAV$

$$\langle u, v\rangle = U^tAV$$

Thus if we identify $V$ with $F^n$ using basis $B$ then bilinear form $<, >$ corresponds to $U^tAV$.

## Matrix of a bilinear form

**Definition 4.**

Let $B = (v_1, ..., v_n)$ be a basis of $V$ and let $\varphi$ be a bilinear form on $V$. The matrix of $\varphi$ with respect to $B$ is

$$[\varphi]_B = \begin{bmatrix} \varphi(v_1, v_2) & \varphi(v_1, v_2) & \cdots & \varphi(v_1, v_n) \\ \varphi(v_2, v_1) & \varphi(v_2, v_2) & \cdots & \varphi(v_2, v_n) \\ \varphi(v_n, v_1) & \varphi(v_n, v_2) & \cdots & \varphi(v_n, v_n) \end{bmatrix}$$

**Expression for a bilinear form on a basis**

**Lemma 1.** Let $B = (v_1, ..., v_n)$ be a basis of $V$ and let $\varphi$ be a bilinear form on $V$. For any $u, v \in V$, we have

$$\varphi(u, v) = U^t AV$$

*Proof.* Let $u = (\alpha_1, ..., \alpha_n)$ and $v = (\beta_1, ..., \beta_n)$. We have

$$\begin{aligned} \varphi(u, v) &= \varphi(\alpha_1 v_1 + ... + \alpha_n v_n, \beta_1 v_1 + ... + \beta_n v_n) \\ &= \textstyle\sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j \varphi(v_i v_j) \\ &= \textstyle\sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j ([\varphi]_C)_{i,j} \\ &= U^T AV \end{aligned}$$

□

If $\varphi$ is a symmetric bilinear form, then

$$\varphi(u, v) = \sum_{1 \leq i \leq n} \alpha_i \beta_i \varphi(e_i, e_i) + 2 \sum_{1 \leq i < j \leq n} \alpha_i \beta_i \varphi(e_i, e_j)$$

**Remark.** $[\varphi]_C$ is the only matrix with this property. A bilinear form $\varphi$ is symmetric if and only if $[\varphi]_C$ is a symmetric matrix.

**Corollary.** Let $V$ be a vector space over a field $F$. Let $B = (v_1, ..., v_n)$ be a basis of $V$. For every $n \times n$ matrix $M$ over $F$, there exists a unique bilinear form $\varphi : V \times V \to F$ such that $\varphi(v_i, v_j) = M_{i,j}$ for $1 \leq i, j \leq n$.

*Proof.* Define $\varphi(u, v) = U^T AV$ and observe that $\varphi$ is bilinear. No other bilinear form with this property exists, since any bilinear form satisfying the assumption has matrix $M$, which Lemma 1. uniquely determines the value of the bilinear form. □

**Example 1.** The bilinear form $\varphi((x_1, y_1), (x_2, y_2)) = x_1x_2 + 2x_1y_2 + 3y_1x_2 + 4y_1y_2$ has matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with respect to the standard basis

$$\varphi((x_1, y_1), (x_2, y_2)) = [x_1, y_1] \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

**Change of basis**

**Lemma 2.** Let $B = (v_1, ..., v_n)$ and $B' = (v'_1, ..., v'_n)$ be two bases of $V$ and let $\varphi$ be a bilinear form on $V$. Let $P = [id]_{B,B'}$. Then

$$[\varphi]_{B'} = P^t[\varphi]_B P$$

*Proof.* We have

$$\begin{aligned} (P^t[\varphi]_B P)_{i,j} &= e_i^t P^t[\varphi]_{B'} P e_j \\ &= [v_i]_B P^t[\varphi]_{B'} P [v_i]_B^t \\ &= [v_i]_{B'}[\varphi]_{B'}[v_j]_{B'}^t \\ &= \varphi(v_i, v_j) = ([\varphi]_{B'})_{i,j} \end{aligned}$$

□

**Application.**

Let $\langle ., . \rangle$ be a bilinear form on $\mathbb{R}^2$ defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 - 3x_1y_2 + x_2y_2$$

1. Find the matrix $A$ of this bilinear form in the basis $\{u_1 = (1, 0)$ and $u_2 = (1, 1)\}$

2. Find the matrix $B$ of given bilinear form in the basis $\{v_1 = (2, 1)$ and $v_2 = (1, -1)\}$

3. Find the transition matrix $P$ from the basis $\{u_i\}$ to $\{v_i\}$ and verify that $B = P^t AP$

**Solution.**

1. Set $A = (a_{ij})$ where $a_{ij} = \langle u_i, u_j \rangle$
   $a_{11} = \langle u_1, u_1 \rangle = \langle (1, 0), (0, 1) \rangle = 2 - 0 + 0 = 2$
   Rest of the entries in the matrix are calculated using the following formula:

$$a_{12} = \langle u_1, u_2 \rangle$$
$$a_{21} = \langle u_2, u_1 \rangle$$
$$a_{22} = \langle u_2, u_2 \rangle$$

Thus the matrix $A$ is as follows

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

2. Similarly matrix $B$ is

$$B = \begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix}$$

3. Now we write $E - 1$ and $v_2$ in terms of $u_1$ and $u_2$

$$(2, 1) = u_1 + u_2$$
$$(1, -1) = 2u_1 - u_2$$

Thus $P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and so $P^t = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$. Thus $P^tAP = P = \begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix} = B$

**Example 2.** Let $A$ be an $n \times n$ matrix in $F$ and define

$$\langle u, v \rangle = U^t AV$$

where $U$ and $V$ are coordinates of $u$ and $v$ respectively in some basis of $V$.
Then we see that this defines a bilinear form on $V$. This coincides with usual inner product of $V$ if $A = I$.

## 1.2 Quadratic forms

**Definition 1.**

A quadratic form $q$ on $V = K^n$ is a function $q : K^n \longrightarrow K$ given by

$$q(u) = q(x_1, x_2, ..., x_n) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j$$

**Property.**

$$\forall c \in K \quad , \quad q(cv) = c^2 q(v)$$

## Symmetric bilinear form on $K^n$ and Quadratic form on $K^n$

**Definition 2.** We can define a quadratic form $q$ using a symmetric bilinear form

$$q(u) = \varphi(u, u)$$

$$q(u) = q\left(x_1, x_2, ..., x_n\right) = \left(x_1, x_2, ..., x_n\right) A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$q(u) = U^t AU$$

**Example.**

Let $q$ be a quadratic form defined on $F^3 : q\left(x, y, z\right) = x^2 + 4xy + 3y^2 - 6yz + xz - 2z^2$. The matrix $A$ of $q$ is

$$A = \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ 2 & 3 & -3 \\ \frac{1}{2} & -3 & -2 \end{bmatrix}$$

**Lemma.**

Let $q$ be a quadratic form on $V = F^n$, $charF \neq 2$, that comes from a symmetric bilinear form $V, q(u) = \varphi(u, u)$. Then the bilinear form may be recovered from $q$:

$$\varphi(u, v) = \frac{1}{2}[q(u + v) - q(u) - q(v)]$$

*Proof.* $\frac{1}{2}(\varphi(u + v, u + v) - \varphi(u, u) - \varphi(v, v)) = \frac{1}{2}(\varphi(u, v) + \varphi(v, u)) = \varphi(u, v)$ □

**Remark.**

The correspondence between quadratic forms and symmetric matrices is one-to-one, when a basis is fixed. So quadratic forms are simply homogeneous polynomials in $n$ variables, where each monomial has a degree 2.

**Definition 2.**

Two matrices $A$ and $B$ are called congruent if $A = P^t BP$ for some non-singular $P$.

**Theorem 1.**

Every real symmetric matrix $A$ is congruent to a diagonal matrix

$$D = P^t AP$$

**Proposition.**

Let $q$ be a quadratic form, $q : V = F^n \longrightarrow F \quad \text{and} \quad dimV = n$

$$q(u) = q(x_1, ..., x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$$

Then $\exists\, \alpha_1, \alpha_2, ..., \alpha_n \in F$ and $l_1, l_2, ..., l_n$ are linear forms such that:

$$q(u) = q(x_1, ..., x_n) = \sum_{i=1}^{n} \alpha_i (l_i(x_1, ..., x_n))^2$$

*Proof.* Using the proof by induction over the dimension of $V$, $dimV = n$

$$\begin{cases} p(1) & \text{is true} \\ p(n-1) \Rightarrow p(n) & \text{is true} \end{cases}$$

For $n = 1, p(1)$ is true.
Let $n \geq 2$:
**Case 1.** $\exists i / a_{ii} \neq 0$, for example, $a_{11} \neq 0$, $q$ has a pure square, the term $a_{11} x_1^2$. Consider all terms which contain $x_1$ and complete the square. We write all terms containing $x_1$ as:

$$a_{11} x_1^2 + \sum_{j=2}^{n} a_{1j} x_1 x_j$$

$$= a_{11} \left[ x_1^2 + 2x_1 \sum_{j=2}^{n} \frac{a_{1j} x_j}{2a_{11}} \right]$$

$$= a_{11} \left[ \left( \underbrace{x_1 + \sum_{j=2}^{n} \frac{a_{1j}}{2a_{11}}}_{l_1} \right)^2 - \left( \sum_{j=2}^{n} \frac{a_{1j} x_j}{2a_{11}} \right)^2 \right]$$

$$= a_{11} \left( l_1(x_1, ..., x_n) \right)^2 - a_{11} \left( \sum_{j=2}^{n} \frac{a_{1j} x_j}{2a_{11}} \right)^2$$

Then $q(x_1,...,x_n) = a_{11}(l_1(x_1,...,x_n))^2 + q'(x_1,...,x_n)$
$q'$ is a quadratic form (Q.F) over $F^{n-1}$ and using the inductive hypothesis, we obtain $q(x_1,...,x_n) = \sum_{k=1}^{n} \alpha_k(l_k(x_1,...,x_n))^2$.

**Case 2.** If $q$ has no terms $a_{ii}x_i^2$, but has a term of the form $a_{ij}x_ix_j$, for example $a_{12}x_1x_2 (a_{12} \neq 0)$. Consider all terms which contain $x_1$ or $x_2$:

$$a_{12}x_1x_2 + \sum_{j=3}^{n} a_{ij}x_ix_j + \sum_{j=3}^{n} a_{2j}x_2x_j$$

$$= a_{12}x_1x_2 + Bx_1 + Cx_2$$

$$a_{12}\left(x_1x_2 + \frac{B}{a_{12}} + \frac{C}{a_{12}}x_2\right)$$

$$= a_{12}\left(\underbrace{x_1 + \frac{C}{a_{12}}}_{a}\right)\left(\underbrace{x_2 + \frac{B}{a_{12}}}_{b}\right) - \frac{BC}{a_{12}^2}$$

$q(u) = a_{12}ab + q''(x_1,...,x_n)$
$q''$ is a Q.F over $F^{n-2}$, since $ab = \frac{1}{4}(a+b)^2 - \frac{1}{4}(a-b)^2$ and by the inductive hypothesis we obtain
$q(u) = \frac{a_{12}}{4}(x_1+x_2+l_1'(x_3,...,x_n))^2 - \frac{a_{12}}{4}(x_1-x_2+l_2'(x_3,...,x_n))^2 + \sum_{k=3}^{n} \alpha_k(l_k(x_3,...,x_n))$
Finally, $q(u) = \sum_{k=1}^{n} \alpha_k(l_k(x_1,...,x_n))^2$. □

**Remark.** This procedure is called Gauss method and is used to write a quadratic form as sum of squares.

**Example.**

$V = \mathbb{R}^4$,

$$q(u) = q(x_1,x_2,x_3,x_4) = x_1^2 + 9x_2^2 + 4x_3^2 + 6x_1x_2 + 4x_1x_3 + 16x_2x_3 + 4x_2x_4 + 8x_3x_4$$

The matrix $A$ of $q$ is $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 9 & 8 & 2 \\ 2 & 8 & 4 & 4 \\ 0 & 2 & 4 & 0 \end{bmatrix}$. $A$ is symmetric because $A^t = A$.
We consider all terms which contain $x_1$:
$x_1^2 + 6x_1x_2 + 4x_1x_3 = x_1^2 + 2x_1(3x_2 + 2x_3) = (x_1 + 3x_2 + 2x_3)^2 - (3x_2 + 2x_3)^2 = (x_1 + 3x_2 + 2x_3)^2 - 9x_2^2 - 4x_3^2 - 12x_2x_3$.

We can write $q$: $q(u) = q(x_1, x_2, x_3, x_4) = (x_1 + 3x_2 + 2x_3)^2 + 4x_2x_3 + 4x_2x_4 + 8x_3x_4$.
We consider all terms which contain $x_2$ or $x_3$ :
$4x_2x_3 + 4x_2x_4 + 8x_3x_4 = 4(x_2x_3 + x_2x_4 + 2x_3x_4) = 4[(x_2 + 2x_4)(x_3 + x_4) - 2x_4^2] = 4[\frac{1}{4}(x_2+x_3+3x_4)^2 - \frac{1}{4}(x_2-x_3+x_4)^2 - 2x_4^2] = (x_2+x_3+3x_4)^2 - (x_2-x_3+x_4)^2 - 8x_4^2$.
Finally,

$$q(u) = q(x_1, x_2, x_3, x_4) = (x_1 + 3x_2 + 2x_3)^2 + (x_2 + x_3 + 3x_4)^2 - (x_2 - x_3 + x_4)^2 - 8x_4^2$$

The matrix $A$ is congruent to a diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -8 \end{bmatrix}$

and we can write $D = P^tAP$. We put $\begin{cases} x'_1 = x_1 + 3x_2 + 3x_3 \\ x'_2 = x_2 + x_3 + 3x_4 \\ x'_3 = x_2 - x_3 + x_4 \\ x'_4 = x_4 \end{cases}$, then $\begin{cases} x_1 = x'_1 - \frac{5}{2}x'_2 - \frac{1}{2}x'_3 + 8x'_4 \\ x_2 = \frac{1}{2}x'_2 + \frac{1}{2}x'_3 - 2x'_4 \\ x_3 = \frac{1}{2}x'_2 - \frac{1}{2}x'_3 - x'_4 \\ x_4 = x'_4 \end{cases}$

We obtain $P = \begin{bmatrix} 1 & -\frac{5}{2} & -\frac{1}{2} & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
The new basis is $B' = (e'_1, e'_2, e'_3, e'_4)$, where
$e'_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $e'_2 = \begin{pmatrix} -\frac{5}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$, $e'_3 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$ and $e'_4 = \begin{pmatrix} 8 \\ -2 \\ -1 \\ 1 \end{pmatrix}$
Finally, $q(u') = q(x'_1, x'_2, x'_3, x'_4) = {x'_1}^2 + {x'_2}^2 - {x'_3}^2 - 8{x'_4}^2$

## Positive definite forms

**Definition 1.**
A bilinear form $\varphi$ on a real vector space $V$ is positive definite, if
$\varphi(u, u) > 0$, for all $u \neq 0$.
A real $n \times n$ matrix $A$ is positive definite, if
$U^tAU > 0$, for all $U \neq 0$.

**Remark.** A bilinear form on $V$ is positive definite if and only if the matrix of the form with respect to some basis of $V$ is positive definite.

**Examples.**

1. A positive definite form on $\mathbb{R}^n$ is given by the dot product (.).
   $u.v = \sum_{i=1}^{n} x_i y_i \Longrightarrow u.u = \sum_{i=1}^{n} x_i^2 > 0$, for all $u = (x_1, x_2, ..., x_n) \neq 0$.

2. Consider the symmetric bilinear form on $\mathbb{R}^n$ which is defined by
   $\varphi(u, v) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$. The quadratic form is
   $q(u) = \varphi(u, u) = x_1^2 - 4x_1x_2 + 5x_2^2 = (x_1 - 2x_2)^2 + x_2^2$.
   Using Gauss method, we can write: $q(u) = (x_1 - 2x_2)^2 + x_2^2$.
   Then the from $\varphi$ is positive definite because $q(u) > 0$, for all $u \neq 0$.

**Tests for positive definiteness**

**Theorem.** The following conditions are equivalent for a symmetric matrix $A$:

**1.** $\varphi(u, u) = U^t AU > 0$ for all $u \neq 0$.

**2.** The eigenvalues of $A$ are all positive $\quad \forall\lambda_i, \lambda_i > 0$.

**3.** One has $detA_k > 0$ for all $k \times k$ upper left submatrices $A_k$ (Sylvester's criterion).

**Remark.** We say that $A$ is negative definite, if $A$ has negative eigenvalues.

**Example 1.** $A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$A_1 = [2] \to detA_1 = 2 > 0$

$A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \to detA_2 = 5 > 0$

$A_3 = A \to detA_3 = detA > 0$

Then $A$ is positive definite.

**Example 2.** Let $a$ be a real parameter and consider the matrix $A = \begin{bmatrix} a & 1 & 1 \\ 1 & 1 & a \\ 1 & a & 5 \end{bmatrix}$

By **Sylvester's criterion**, $A$ is positive definite if and only if

$$a > 0, \qquad det \begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix} > 0, \qquad detA > 0$$

The first two conditions give $a > 0$ and $a > 1$, while

$$detA = -a^3 + 7a - 6 = -(a-1)(a-2)(a+3)$$

It easily follows that $A$ is positive definite if and only if $1 < a < 2$.

## Orthogonality

Suppose that $\varphi$ is a symmetric bilinear form on a real vector space $V$:

1. **Orthogonal vectors:** Two vector $u, v$ are called orthogonal, if $\varphi(u, v) = 0$.
2. **Orthogonal basis:** A basis $B = (v_1, v_2, ..., v_n)$ of $V$ is called orthogonal, if $\varphi(v_i, v_j) = 0$ for all $i \neq j$ and it is called orthonormal, if it is orthogonal with $\varphi(v_i, v_i) = 1$ for all $i$.
3. If $F$ is a subspace of $V$, the orthogonal of $F$ is
$F^{\perp} = \{u \in V/\varphi(u, v) = 0, \forall v \in F\}$, which is also a subspace of $V$.
4. **Isotropic vectors:** A vector $u(u \neq 0)$ is called isotropic, if $q(u) = \varphi(u, u) = 0$.
5. **Kernel, non-degenerate forms:** $\varphi$ is called a non-degenerate form, if
$E^{\perp} = \{u \in V/\varphi(u, v) = 0, \forall v \in V\} = \{0\}$. Otherwise, $\varphi$ is called degenerate.
The kernel of $\varphi$ or $q$, $ker\varphi = kerq = E^{\perp}$.
6. The isotropic cone of a quadratic form $q$ is the set of all isotrops of $V$ under $q$.
$C(q) = \{u \in V/q(u) = 0\}$
7. A subspace $F$ of $V$ is called isotropic, if $F \cap F^{\perp} \neq (0)$.

**Proprieties.**

- $kerq \subset C(q)$
- $dimV = dimker(q) + rg(q)$
- $dimV = dimF + dimF^{\perp} - dim(F \cap kerq)$, $F$ is a subspace of $V$.
In particular, if $q$ is non-degenerate, $dimV = dimF + dimF^{\perp}$.
- $F^{\perp\perp} = F + kerq$
- $V = F \oplus F^{\perp} \iff F$ is not isotropic ($F \cap F^{\perp} = 0$).

## Gram-Schmidt procedure

Suppose that $(v_1, v_2, ..., v_n)$ is a basis of a dot product space $V$, then we can find an orthogonal basis $(v'_1, v'_2, ..., v'_n)$ as follows:
We put
$v'_1 = v_1$
$v'_2 = v_2 - \frac{v_2.v'_1}{v'_1.v'_1} v'_1$

.
.
.
$v'_n = v_n - \sum_{i=1}^{n-1} \frac{v_n.v'_i}{v'_i.v'_i} v'_i$
then $v'_1, v'_2, ..., v'_n$ are orthogonal.

**Example.**

We find an orthogonal basis of $\mathbb{R}^3$, starting with the basis
$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
We define the first vector by $v'_1 = v_1$ and the second by
$v'_2 = v_2 - \frac{v_2.v'_1}{v'_1.v'_1} v'_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
Then $v'_1, v'_2$ are orthogonal and we may define the third vector by
$v'_3 = v_3 - \frac{v_3.v'_1}{v'_1.v'_1} v'_1 - \frac{v_3.v'_2}{v'_2.v'_2} v'_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

**Theorem.**

The eigenvalues of a real symmetric matrix $A$ are all real. i.e $\lambda_i \in \mathbb{R}$
The eigenvectors of a real symmetric matrix $A$ corresponding to distinct eigenvalues are necessarily orthogonal to one another. i.e $\lambda_i \neq \lambda_j \Rightarrow v_i.v_j = 0$

## Orthogonal matrices

**Definition.**

A real $n \times n$ matrix $P$ is called orthogonal, if $P^t P = I_n$ i.e $P^{-1} = P^t$.

**Proprieties.**

- To say that an $n \times n$ matrix is orthogonal is to say that the columns of $P$ form an orthonormal basis of $\mathbb{R}^n$.
- The product of two $n \times n$ orthogonal matrices is orthogonal.

**Example.** $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

## Spectral theorem

Every real symmetric matrix $A$ is diagonalisable. In fact, there exists an orthogonal matrix $P$ such that $P^{-1}AP = P^tAP$ is diagonal.

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = P^{-1}AP = P^tAP$$

**Remarks.**

- When the eigenvalues of $A$ are distinct, the eigenvectors of $A$ are orthogonal and we may simply divide each of them by its norm to obtain an orthonormal basis of $\mathbb{R}^n$.

- When the eigenvalues of $A$ are not distinct, the eigenvectors of $A$ may not be orthogonal. In that case, one may use the **Gram-Schmidt procedure** to replace eigenvectors that have the same eigenvalue with orthogonal eigenvectors that have the same eigenvalue.

- The converse of the spectral theorem is also true. That is, if $P$ is an orthogonal matrix and $P^tAP$ is diagonal, then $A$ is symmetric.

## Diagonalisation of quadratic forms

**Theorem.**

Let $q(u) = U^tAU$ for some symmetric $n \times n$ matrix $A$. Then there exists an orthogonal change of variables $U = PU'$ such that:

$$q(u') = q(x'_1, x'_2, ..., x'_n) = \sum_{i=1}^{n} \lambda_i {x'_i}^2$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of the matrix $A$.

### Signature of a quadratic form

**Definition.** The signature of a quadratic form $q(u) = U'AU$ is defined as the pair of integers $(n_+, n_-)$, where $n_+$ is the number of positive eigenvalues of $A$ and $n_-$ is the number of negative eigenvalues of $A$.

**Examples.**

**1.** We diagonalise the quadratic form in $\mathbb{R}^2$, $B = (e_1, e_2)$ the standard basis

$$q(u) = q(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 2x_2^2$$

We have $A = M_B(q) = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$.
The eigenvalues $\lambda = 1, 6$ are distinct and one can easily check that
$P = (e'_1 e'_2) = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$, then $D = M_{B'}(q) = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} = P^{-1}AP = P^tAP$
As usual, the columns of $P$ were obtained by finding the eigenvectors of $A$ and by dividing each eigenvector by its norm.
Changing variables by $U = PU'$, we now get $U' = P^tU$ and also

$$\boxed{q(u') = q(x'_1, x'_2) = {x'_1}^2 + 6{x'_2}^2 = \left(\frac{x_1 - 2x_2}{\sqrt{5}}\right)^2 + 6\left(\frac{2x_1 + x_2}{\sqrt{5}}\right)^2}.$$

We can use the Gauss method to find the sum of squares of $q$.
We take $(I) : 5x_1^2 + 4x_1x_2 = 5(x_1^2 + \frac{4}{5}x_1x_2)$
$= 5[x_1^2 + 2x_1(\frac{2}{5}x_2)] = 5[(x_1 + \frac{2}{5}x_2)^2 - (\frac{2}{5}x_2)^2] = 5[(x_1 + \frac{2}{5}x_2)^2 - \frac{4}{25}x_2^4] = 5(x_1 + \frac{2}{5}x_2)^2 - \frac{4}{5}x_2^2$
replace this in $q(u)$
$q(u) = 5(x_1 + \frac{5}{2}x_2)^2 - \frac{4}{5}x_2^2 + 2x_2^2$, we obtain

$$\boxed{q(u) = 5(x_1 + \tfrac{2}{5}x_2)^2 + \tfrac{6}{5}x_2^2}$$

We put $\begin{cases} x'_1 = x_1 + \frac{5}{2}x_2 \\ x'_2 = x_2 \end{cases} \Rightarrow \begin{cases} x_1 = x'_1 - \frac{5}{2}x'_2 \\ x_2 = x'_2 \end{cases}$
We obtain the orthogonal matrix $P$ and the new basis $B' = (e'_1, e'_2)$:
$P = (e'_1 e'_2) = \begin{bmatrix} 1 & -\frac{5}{2} \\ 0 & 1 \end{bmatrix}$ and the formula $D = M_{B'}(q) = \begin{bmatrix} 5 & 0 \\ 0 & \frac{6}{5} \end{bmatrix} = P^tAP$.
Finally,

$$\boxed{q(u') = q(x'_1, x'_2) = 5{x'_1}^2 + \tfrac{6}{5}{x'_2}^2}$$

The signature of $q$ is $(2, 0)$, $q$ is a non-degenerate form and the rank of $q$ is 2.

**2.** We consider the quadratic form defined in $\mathbb{R}^3$ by the real symmetric matrix
$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$q(u) = U^t A U = (x, y, z)\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x, y, z)\begin{bmatrix} 2x + y + z \\ x + 2y + z \\ x + y + 2z \end{bmatrix}$$

$$= x(2x + y + z) + y(x + 2y + z) + z(x + y + 2z)$$
$$= 2x^2 + xy + xz + yx + 2y^2 + 2yz + zx + zy + 2z^2$$

$$\boxed{q(u) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz}$$

**Diagonalisation of $A$:**

We have $P(\lambda) = -(\lambda - 1)^2(\lambda - 4)$, $\quad P(\lambda) = 0 \Rightarrow \begin{cases} \lambda_1 = 1, m_1 = 2 \\ \lambda_2 = 4, m_2 = 1 \end{cases}$

$E(\lambda_1) = span\{v_1, v_2\}$, $E(\lambda_2) = span\{v_3\}$

where $v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In this case, use the Gram-Schmidt procedure to replace $v_1, v_2$ by two orthogonal eigenvectors $v_1', v_2'$, dividing each of $v_1', v_2', v_3$ by its norm, we then obtain he columns of the orthogonal matrix: we put

$v_1' = v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$v_2' = v_2 - \frac{v_2.v_1'}{v_1'.v_1'} v_1' = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$

$v_3' = v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

We obtain the orthogonal matrix and the new basis (orthonormal basis)
$B' = (e_1', e_2', e_3')$ and

$P = (e_1' e_2' e_3') = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

We obtain the formula $D = M_{B'}(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = P^{-1}AP = P^tAP$

we have $U = PU'$, then $U' = P^tU \Rightarrow \begin{cases} x'_1 = \frac{1}{\sqrt{2}}(-x_1 + x_3) \\ x'_2 = \frac{1}{\sqrt{6}}(-x_1 + 2x_2 - x_3) \\ x'_3 = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3) \end{cases}$

Finally,

$$\boxed{q(u') = (x'_1, x'_2, x'_3) = {x'_1}^2 + {x'_2}^2 + 4{x'_3}^2}$$

The signature of $q$ is $(3, 0)$ and the rank equals 3. $q$ is a non-degenerate form.

**Gauss Method:**

$q(u) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz$

$$\begin{aligned} \text{We take } (I):\ 2x^2 + 2xy + 2xz &= 2\left[x^2 + 2x\left(\frac{y+z}{2}\right)\right] \\ &= 2\left[\left(x + \frac{1}{2}y + \frac{1}{2}z\right)^2 - \left(\frac{y+z}{2}\right)^2\right] \\ &= 2\left(x + \frac{1}{2}y + \frac{1}{2}z\right)^2 - \frac{1}{2}\left(y^2 + z^2 + 2yz\right) \\ &= 2\left(x + \frac{1}{2}y + \frac{1}{2}z\right)^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 - yz \end{aligned}$$

$q(u) = 2(x + \frac{1}{2}y + \frac{1}{2}z)^2 + \frac{3}{2}y^2 + \frac{3}{2}z^2 + yz$

$$\begin{aligned} \text{We take } (II):\ \frac{3}{2}y^2 + yz &= \frac{3}{2}\left(y^2 + \frac{2}{3}yz\right) = \frac{3}{2}\left[y^2 + 2y\left(\frac{z}{3}\right)\right] \\ &= \frac{3}{2}\left[\left(y + \frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^2\right] \\ &= \frac{3}{2}\left(y + \frac{1}{3}z\right)^2 - \frac{1}{6}z^2 \end{aligned}$$

Then,

$$\boxed{q(u) = q(x, y, z) = 2(x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 + \tfrac{3}{2}(y + \tfrac{1}{3}z)^2 + \tfrac{4}{3}z^2}$$

$$D = M_{B'}(q) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = P^t A P$$

We put $\begin{cases} x' = x + \frac{1}{2}y + \frac{1}{2}z \\ y' = y + \frac{1}{3}z \\ z' = z \end{cases} \Rightarrow \begin{cases} x = x' - \frac{1}{2}(y' - \frac{1}{3}z') - \frac{1}{2}z' \\ y = y' - \frac{1}{3}z' \\ z = z' \end{cases} \Rightarrow \begin{cases} x = x' - \frac{1}{2}y' - \frac{1}{3}z' \\ y = y' - \frac{1}{3}z' \\ z = z' \end{cases}$

We obtain the orthogonal matrix $P = (e'_1 e'_2 e'_3) = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$

The new basis is $B'(e'_1, e'_2, e'_3)$: $e'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e'_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, e'_3 = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$

Finally,

$$\boxed{q(u') = q(x', y', z') = 2x'^2 + \frac{3}{2}y'^2 + \frac{4}{3}z'^2}$$

# Chapter 2

# Hermitian and hermitian quadratic forms

Let $V$ be a $\mathbb{C}-$vector space.

**Definition 1.**

A hermitian form is a function $\varphi$ of $V$ in $\mathbb{C}$, satisfying

$$\begin{gathered}\varphi(u_1+u_2,v)=\varphi(u_1,v)+\varphi(u_2,v)\\ \varphi(u,v_1+v_2)=\varphi(u,v_1)+\varphi(u,v_2)\\ \varphi(\alpha u,v)=\alpha\varphi(u,v)\\ \varphi(u,\alpha v)=\bar{\alpha}\varphi(u,v)\\ \varphi(u,u)=\overline{\varphi(u,u)}\end{gathered}$$

**Remark.**

Since $\varphi(u,u)=\overline{\varphi(u,u)}$, then $\varphi(u,u)\in\mathbb{R}$.

**Definition 2.**

An hermitian quadratic form is a function $q:\ V\to\mathbb{R}$ given by

$$q(u)=\varphi(u,u)$$

**Propriety.**

$$q(\alpha u)=|\alpha|^2q(u),\ \text{ for all }\ \alpha\in\mathbb{C}$$

**Proposition.**

Let $\varphi$ be an hermitian form and $q$ is the associated hermitian quadratic form of $\varphi$, then

$$Re(\varphi(u,v)) = \frac{q(u+v)-q(u)-q(v)}{2} = \frac{q(u+v)-q(u-v)}{4}$$

$$Im(\varphi(u,v)) = \frac{q(u+iv)-q(u)-q(v)}{2} = \frac{q(u+iv)-q(u-iv)}{4}$$

and,

$$\varphi(u,v) = \frac{q(u+v)-q(u-v)+iq(u+iv)-iq(u-iv)}{4}$$

**Examples.**

1. The form $z \mapsto |z|^2$ is an hermitian quadratic form on $V = \mathbb{C}$, associated with the hermitian form
$$(z,w) \mapsto z\overline{w}$$

2. The form $(z_1,...,z_n) \mapsto |z_1|^2 + ... + |z_n|^2$ is an hermitian quadratic form, associated with the hermitian form
$$((z_1,...,z_n),(w_1,...,w_n)) \mapsto z_1\bar{w_1} + ... + z_n\bar{w_n}$$

**Definition 3.**

Let $A \in M_n(\mathbb{C})$. $A$ is called an hermitian matrix if $A^t = \bar{A}$.
If $A = (a_{ij})_{1\leq ij\leq n}$, then $a_{ij} = \overline{a_{ij}}$ for all $i,j$.

**Proposition.**

Let $A \in M_n(\mathbb{C})$, then $\varphi : \begin{cases} \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C} \\ (u,v) \mapsto U^tA\overline{V} \end{cases}$ is an hermitian form over $\mathbb{C}^n$.

**Proposition-Definition.**

Let $\varphi$ be an hermitian form and $q$ is the associated hermitian quadratic form of $\varphi$. The matrix $A = (a_{ij})_{1\leq ij\leq n}$ is a matrix of $\varphi$ (or of $q$) over the standard basis $B = (e_1, e_2, ..., e_n)$, where
$$a_{ij} = \varphi(e_i, e_j)$$

1. The matrix $A$ is hermitian.

**2.** Let $u = \sum_{i=1}^{n} x_i e_i$ and $v = \sum_{j=1}^{n} y_j e_j$, then

$$\varphi(u,v) = \sum_{1 \leq ij \leq n} a_{ij} x_i \overline{y_j} = U^t A \overline{V}$$

**3.** If $B' = (e'_1, e'_2, ..., e'_n)$ is another basis of $V$, then

$$A' = M_{B'}(\varphi) = P^t A \overline{P}$$

**Remark.**

We have

$$q(u) = \sum_{i=1}^{n} a_{ii} |x_i|^2 + \sum_{1 \leq i < j \leq n} 2Re(a_{ij} x_i \overline{x_j})$$

The rank of $\varphi$ is the rank of its matrix over all basis of $V$.
The form $\varphi$ (or $q$) is called non-degenerate if $\varphi$ is of the rank $n$.

**Theorem.**

There exists an orthogonal basis of $V$ for the hermitian quadratic form $q$ and

$$q(u) = q(x_1, x_2, ..., x_n) = \sum_{i=1}^{k} \alpha_i |l_i(x_1, x_2, ..., x_n)|^2$$

where $\alpha_1, ..., \alpha_k \in \mathbb{R}$ and $l_1, ..., l_k$ are the linear forms over $V$.

**Example.**

Let $q$ be an hermitian quadratic form over $\mathbb{C}^3$ defined by

$$q(z_1, z_2, z_3) \mapsto z_1\overline{z_1} + 3z_2\overline{z_2} - z_3\overline{z_3} + iz_1\overline{z_2} - iz_2\overline{z_1} - z_1\overline{z_3} - z_3\overline{z_1} + 2iz_2\overline{z_3} - 2iz_3\overline{z_2}$$

The matrix of $q$ is $A = M_B(q) = \begin{bmatrix} 1 & i & -1 \\ -i & 3 & 2i \\ -1 & -2i & 1 \end{bmatrix}$

$$\begin{aligned} q(z_1, z_2, z_3) &= \big(|z_1|^2 + 2Re(iz_1\overline{z_2}) + 2Re(-z_1\overline{z_3})\big) + 3|z_2|^2 - |z_3|^2 + 2Re(2iz_2\overline{z_3}) \\ &= \big(|z_1 - iz_2 - z_3|^2 - |z_2|^2 - |z_3|^2 - 2Re(iz_2\overline{z_3})\big) + 3|z_2|^2 - |z_3|^2 + 4Re(iz_2\overline{z_3}) \\ &= |z_1 - iz_2 - z_3|^2 + 2\left|z_2 - \frac{iz_3}{2}\right|^2 - \frac{5|z_3|^2}{2} \end{aligned}$$

$$
\begin{aligned}
q(z_1, z_2, z_3) &= 2|w_1|^2 - 2|w_2|^2 + 2Re((1+2i)w_1\overline{z_3}) - 2Re(w_3\overline{z_3}) \\
&= 2\left|w_1 + \left(\frac{1}{2} - i\right) z_3\right|^2 - \frac{5}{2}|z_3|^2 - 2\left|w_2 + \frac{z_3}{2}\right|^2 + \frac{|z_3|^2}{2} \\
&= \frac{1}{2}|iz_1 + z_2 + (1-2i)z_3|^2 - \frac{1}{2}|iz_1 - z_2 + z_3|^2 - 2|z_3|^2
\end{aligned}
$$