Worksheet N°2

Summary: First-order differential equations

The type	the form
(DES) Differential equation with separable variables	$y'f(y) = g(x) \iff \boxed{y' = \frac{g(x)}{f(y)}}$
(HDE) Homogeneous differential equations	$\boxed{y' = F\left(\frac{y}{x}\right)}$
(LHDE 01) Linear homogeneous differential equations of order 01	$y' + f(x)y = 0 \iff \boxed{y' = h(x)y}$
(LDE 01) Linear differential equation of order 01	$y' + f(x)y = g(x) \iff \boxed{y' = h(x)y + g(x)}$
(BDE) Bernoulli differential equation	$y' + f(x)y = g(x)y^{\alpha} \iff \boxed{y' = h(x)y + g(x)y^{\alpha}}, \alpha \in \mathbb{R} - \{0;1\}$

Exercise 01:

Give the type of the following differential equations (without solving them)

① $\ln(y)y' - e^x = 0$

② $(x-y)ydx - x^2dy = 0$

③ $(x - \sin(x))y' = (1 - \cos(x))\, y$

④ $y' + \tan(x)y - \sin(x) = 0$

⑤ $-y' + \tan(x)y - \sin(x)y^2 = 0$

Exercise 02:(Differential equation with separable variables)

Solve the following differential equations

① $xy' = y$

② $(x^2+1)y' = y^2 + 1$

③ $xy' = y\ln(y)$

Exercise 03: (Linear homogeneous differential equations of order 01)

Solve the following differential equations

① $(1-x^2)y' - 2xy = x$

② $xy' - 2y + x^3e^{-x} = 0$

③ $xy' + y\tan(x) - \dfrac{1}{\cos(x)} = 0$

④ $y' + ay = e^{-x}$

Exercise 04: Homogeneous differential equations

Solve the following differential equations

① $(x-y)ydx - x^2dy = 0$

② $xy' = y + x\cos\left(\dfrac{y}{x}\right)$

③ $xy' = y - x$

Exercise 05 (Equations of Bernoulli)

Solve the following differential equations

① $y' + \dfrac{1}{x}y + y^2 = 0.$ | ② $xy' + y - xy^3 = 0.$ | ③ $y' - \dfrac{x}{2}y = \sqrt{y}x.$

Exercise 06:

We consider the first-order differential equation

$$y' + y\tan(x) = \sin(x)\cos(x) \quad \text{(E)}$$

① Solve the homogeneous equation (without right-hand side) associated with (E)

② Using the method of variation of consonants, find a particular solution of (E), then give the set of all solutions of (E).

③ Calculate the solution of (E) satisfying $y\left(\frac{\pi}{4}\right) = 0$.

④ Deduce the solution of the following equation (EDB)."

$$-z' + z\tan(x) = \sin(x)\cos(x)z^2 \quad \text{(EDB)}$$

⑤ Solve the following differential equation

$$(x-y)\,ydx - x^2dy = 0$$

Exercise 07 (Exam 2015-University of A.Mira-Béjaia)

① Calculate the indefinite integral $\displaystyle\int \frac{9}{x^2-5x-14}dx$

② Deduce the value of the definite integral $\int_0^1 \frac{9}{x^2-5x-14}dx$.

③ By the change of variable $t=\sin(x)$, calculate: $I=\int_0^{\frac{\pi}{2}} \frac{9\cos(x)}{-14-5\sin(x)+\sin^2(x)}dx$

④ Let $x\in]7;+\infty[$, Solve the following differential equation:

$$y'-\frac{9}{x^2-5x-14}y=\frac{x-7}{x^2-5x-14} \tag{E}$$

Exercise 08 :

① Calculate the integral

$$\int \frac{2\ln(x)}{x(1+\ln^2(x))}dx$$

.

② Resolve on $I=]0,+\infty[$ the equation

$$x\left(1+\ln^2(x)\right)y'+2\ln(x)y=1$$

Exercise 09

$I)$ We consider the first-order differential equation.

$$y'-\left(2x-\frac{1}{x}\right)y=1 \tag{E}$$

① Solve the homogeneous equation (EH) (without a right-hand side) associated with (E)

$$y'-\left(2x-\frac{1}{x}\right)y=0 \tag{EH}$$

② Using the method of variation of consonants, find **a particular solution** y_p de (E), of (E), and then give the set of all solutions to (E).

Exercise 10 : (Exam 2016-2017 University of A.Mira-Béjaia)

① Calculate $I=\int \frac{2x+1}{x^2(x+1)}dx$ et $K=\int \frac{1}{x^2}\ln(x^2+x)dx$.

② Solve the following differential equation

$$y'-\frac{2}{x}y=\ln(x^2+x)$$

Exercise 11 : (Exam 2010-2011 University of A.Mira-Béjaia)

Let f be a function defined by

$$f(x)=\frac{1}{x(1-x^2)},\quad x\in\mathbb{R}-\{-1;0;1\}$$

① Calculer $\int f(x)dx$.

② Resolve the following differential equation

$$y' - y = \frac{e^x}{x(1-x^2)} \tag{E}$$

Exercise 12 (devoire) : (Exam 2011 University of A.Mira-Béjaia)

We consider the first-order differential equation.

$$y' + 2y = 3e^{-2x} \tag{E}$$

① Check that $y_p = 3xe^{-2x}$is a particular solution of (E).

② Solve the homogeneous equation (EH) associated with (E).

$$y' + 2y = 0 \tag{EH}$$

③ Deduce the solutions of (E)

Exercise 13 : (Exam 2018 University of M'sila)

We consider the first-order differential equation

$$y' + 4y = \sin(3x)e^{-4x} \tag{E}$$

① Solve the homogeneous equation (EH) (without a right-hand side) associated with (E).

$$y' + 4y = 0 \tag{EH}$$

② Using the method of variation of consonants, find **a particular solution** y_p of (E), then give the set of all solutions of (E).

③ Calculate the solution y_1 of (E) satisfying $y_1(\pi) = 0$

Exercise 14

We consider the first-order differential equation.

$$y' + y = \frac{e^{-x}}{1+x^2} \tag{E}$$

① Solve the homogeneous equation (EH) (without a right-hand side) associated with (E).

② Using the method of variation of consonants, find **a particular solution** y_p of (E), then give the set of all solutions of (E).

Exercise 15 : (Exam 2018 University de A.Mira-Béjaia)

We consider the first-order differential equation.

$$y' + (3x^2+1)y = x^2e^{-x} \tag{E}$$

① Solve the homogeneous equation (without a right-hand side) associated with (E)

② Using the method of variation of consonants, find **a particular solution** y_p of (E), then give the set of all solutions of (E).

. ③ Calculate the solution y_1 of (E) satisfying $y_1(\pi) = 0$

Exercise 16 (devoire) : (Exam 2016 University of A.Mira-Béjaia)

We consider the first-order differential equation.

$$y' - 2y = 4 - x \tag{E}$$

① Solve the homogeneous equation (without a right-hand side) associated with (E)

② Check that $y_p = \frac{1}{2}x - \frac{7}{4}$ is a particular solution of (E).

③ Give the set of all solutions of (E).

④ Calculate the solution y_1 of (E) satisfying $y_1(0) = 1$

Exercise 17 : (Exam 2016 University of M'sila)

① Calculate the integral

$$\int \frac{x}{\sin^2 x} dx$$

.

② Solve on $I = \left]0, \frac{\pi}{2}\right[$ the equation

$$y' \sin(x) - y\cos(x) = x$$

Exercise 18 : (Exam 2015 University of A.Mira-Béjaia)

We consider the first-order differential equation

$$2y' - y = \cos(x) \tag{E}$$

① Solve the homogeneous equation associated with (E)

① Check that $y_p = -\frac{1}{5}\cos(x) + \frac{2}{5}\sin(x)$is a particular solution of (E).

③ Deduce the general solution of (E).

③ Calculate the solution y_1 of (E) satisfying $y_1(0) = 0$

Exercise 19 (Exam 2020 University of M'sila) (6pts)

Tick the correct answer for each question.

① The value of the integral $\int_{\frac{\pi}{2}}^{\pi} \pi \sin(10x)dx$ is **(2pts)**

a) $\frac{\pi}{10}$ ☐.   b) $-\frac{\pi}{5}$ ☐.   c) $\frac{\pi}{5}$ ☐.

② the equation $x^2y' + xy = y^2 + 4x^2$ is a differential equation of **(2pts)**

a) Bernoulli ☐.   **b)** homogeneous. ☐.   **c)** à separable variables. ☐.

③ A particular solution of the equation $2y' - y = \cos(x)$ is **(2pts)**

a) $y_p = -\frac{1}{5}\cos(x) + \frac{2}{5}\sin(x)$ ☐.   b) $y_p = \frac{1}{5}\cos(x) - \frac{2}{5}\sin(x)$ ☐.

Exercice 20 :(Exam 2020 University of M'sila) (6pts)

We consider the first-order differential equation

$$y' - y = e^x \sin(x) \tag{E}$$

① Solve the homogeneous equation (without a right-hand side) associated with (E)

② Using the method of variation of consonants, find **a particular solution** y_p of (E), then give the set of all solutions of (E).

③ Calculate the solution y_1 of (E) satisfying $y_1(\pi) = 0$

Worksheet N°3

Summary: Linear second-order differential equations with constant coefficients

① A linear second-order differential equation with constant coefficients is an equation of the form

$$ay'' + by' + cy = f(x) \tag{E}$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$, and f is a continuous function on an open interval I.
②The equation

$$\boxed{ay'' + by' + cy = 0} \tag{EH}$$

is called the associated homogeneous equation (EH) to (E).
③ The equation

$$\boxed{ar^2 + br + c = 0} \tag{EC}$$

is called the characteristic equation associated to (EH).
③ The general solutions of the equation (E) are obtained by adding the general solutions of the homogeneous equation (EH) to a particular solution of (E).i.e;

$$\boxed{y_G = y_h + y_p}$$

How to calculate y_h?

Let y_h be the solution of the equation: $ay'' + by' + cy = 0$ (EH). And let $ar^2 + br + c = 0$ (EC) be the associated characteristic equation to (EH), and $\Delta = b^2 - 4ac$ the discriminant of the equation (EC):

	The solutions of $ar^2 + br + c = 0$ are	The solutions of $ay'' + by' + cy = 0$ are
$\Delta > 0$	$r_1 = \frac{-b + \sqrt{\Delta}}{2a}; r_2 = \frac{-b - \sqrt{\Delta}}{2a}$	$\boxed{y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$; ou $c_1, c_2 \in \mathbb{R}$
$\Delta = 0$	$r = \frac{-b}{2a}$	$\boxed{y_h = (c_1 + c_2 x)e^{rx}}$; ou $c_1, c_2 \in \mathbb{R}$
$\Delta < 0$	$r_1 = \alpha + i\beta; r_2 = \overline{r_1} = \alpha - i\beta$	$\boxed{y_h = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)}$; ou $c_1, c_2 \in \mathbb{R}$

Search for a particular solution y_p

We provide four (04) important particular cases and a general method.

Method 01:

Right-hand side of the type	Roots of the characteristic equation	A particular solution of
$f(x) = P_n(x)$, where P_n is a polynomial of degree n	The number 0 is not a root of the characteristic equation	$y_p = Q_n(x)$ where Q_n is a polynomial of degree n
	The number 0 is a root of multiplicity k of the characteristic equation	$y_p = x^k Q_n(x)$ where Q_n is a polynomial of degree n
$f(x) = P_n(x)e^{\alpha x}$, where P_n is a polynomial of degree n $\alpha \in \mathbb{R}$	The number α is not a root of the characteristic equation	$y_p = Q_n(x)e^{\alpha x}$ where Q_n is a polynomial of degree n
	The number α is a root of multiplicity k of the characteristic equation	$y_p = x^k Q_n(x)e^{\alpha}$ where Q_n is a polynomial of degree n
$f(x) = P_1(x)\cos(\beta x) + P_2(x)\sin(\beta x)$, where $\beta \in \mathbb{R}$ and P_1, P_2 two polynomials,	the numbers$\pm i\beta$ are not roots of the characteristic equation	$y_p = Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)$ where Q_1 and Q_2 are two polynomials of degree $n = \max(deg\{P_1, P_2\})$
	the numbers $\pm i\beta$ are roots of multiplicity k of the characteristic equation	$y_p = x^k(Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x))$ where Q_1 and Q_2 are two polynomials of degree $n = \max(deg\{P_1, P_2\})$
$f(x) = e^{\alpha x}(P_1(x)\cos(\beta x)$ $+P_2(x)\sin(\beta x))$, where $\alpha, \beta \in \mathbb{R}$ and P_1, P_2 two polynomials,	The number $\alpha \pm i\beta$ is not a root of the characteristic equation	$y_p = e^{\alpha x}(Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x))$ where Q_1 and Q_2 are two polynomials of degree $n = \max(deg\{P_1, P_2\})$
	The number $\alpha \pm i\beta$ is a root of multiplicity k of the characteristic equation	$y_p = x^k e^{\alpha x}(Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x))$ where Q_1 and Q_2 are two polynomials of degree $n = \max(deg\{P_1, P_2\})$

Method 02: Variation of constants.
If $\{y_1, y_2\}$ is a set of solutions of the homogeneous equation (EH), we seek a particular solution in the form $y_p = c_1(x)y_1 + c_2(x)y_2$, such that $c_1(x)$ and $c_2(x)$ are two functions satisfying :

$$\begin{cases} c_1'(x)y_1 + c_2'(x)y_2 = 0 \\ c_1'(x)y_1' + c_2'(x)y_2' = \dfrac{f(x)}{a} \end{cases}$$

Example

Solve the following differential equations :
$y'' - 5y' + 6y = 0 ...(1)$ $\quad y'' - 6y' + 9y = 0...(2)$ $\quad y'' - 2y' + 5y = 0...(3)$

Solution

1. The associated characteristic equation to $y'' - 5y' + 6y = 0$ is $r^2 - 5r + 6 = 0$, which has two solutions: $r_1 = 2$ and $r_2 = 3$. Therefore, the solutions are the functions defined on $\mathbb{R}$ by $y(x) = c_1e^{2x} + c_2e^{3x}$ where $c_1, c_2 \in \mathbb{R}$.

2. The associated characteristic equation to $y'' - 6y' + 9y = 0$ is $r^2 - 6r + 9 = 0$, which has a double root: $r = 3$. Therefore, the solutions are the functions defined on $\mathbb{R}$ by
$y(x) = (c_1 + c_2x)e^{3x}$ where $c_1, c_2 \in \mathbb{R}$

3. The associated characteristic equation to $y'' - 2y' + 5y = 0$ is $r^2 - 2r + 5 = 0$, which has two complex solutions: $r_1 = 1 + 2i$ and $r_2 = 1 - 2i$. Therefore, the solutions are the functions defined on $\mathbb{R}$ by $y(x) = e^x \left(c_1 \cos(2x) + c_2 \sin(2x)\right)$ where $c_1, c_2 \in \mathbb{R}$

Exercise 01 :

Solve the following differential equations

1) $y'' - y' - 2y = 0$ | 2) $y'' - 4y' + 4y = 0$ | 3) $y'' - 6y' + 10y = 0$

Exercise 03 :

Solve the following differential equations

1) $y'' - 3y' + 2y = 0$ | 2) $y'' - 2y' + y = 0$ | 3) $y'' + 9y = 0$

Exercise 02:

Solve the following differential equations:

1) $y'' - 3y' + 2y = x^2 + 1$
2) $y'' - 3y' = 3x$
3) $y'' - 5y' + 6y = 2xe^x$
4) $y'' - 5y' + 6y = 2xe^{3x}$
5) $y'' - 2y' + y = 2e^x$
6) $y'' - 5y' + 6y = 2xe^{3x} + 2xe^x$
7) $y'' + 4y = 3\sin(2x)$
8) $y'' + 4y = 3\sin(2x) + 5x\cos(2x)$
9) $y'' + 4y = 3\sin(2x) + 5x\cos(3x)$
10) $y'' - 4y' + 5y = \sin(x)e^{2x}$

Exercise 04 :

Solve the following differential equations :

1) $y'' - 7y' + 12y = 4x^2$
2) $y'' - 7y' = 2x$
3) $y'' - 7y' + 12y = 3xe^{2x}$
4) $y'' - 7y' + 12y = 2xe^{4x}$
5) $y'' + 4y' + 4y = e^{-2x}$
6) $y'' - 7y' + 12y = 2xe^{4x} + 4x^2$
7) $y'' + 9y = 2\cos(3x) + x\sin(3x)$
8) $y'' + 9y = 2\cos(3x) + x\sin(2x)$
9) $y'' - 2y' + 5y = \sin(2x)e^x$

Exercise 06 : (Exam 2011 University of A.Mira-Béjaia)

We consider the following second-order differential equation

$$y'' - 3y' + 2y = xe^{2x} + \cos^2(x) - \sin^2(x)...(E)$$

① Resolve the homogeneous equation associated with (E)

② Give the particular solution of the equation:

$$y'' - 3y' + 2y = xe^{2x}...(E_1)$$

③ Give the particular solution of the equation:

$$y'' - 3y' + 2y = \cos(2x)...(E_2)$$

④ Deduce the particular solution of the equation (E)

⑤ Give the general solution of the equation (E)

Exercise 07 :

Résoudre l'équation suivante, sur l'intervalle $\left]-\frac{\pi}{2}; +\frac{\pi}{2}\right[$

$$y'' + y = \frac{1}{\cos(x)}$$

Exercise 08 (Exam 2017-University of M'sila)(6pts) (*)

① Solve the following differential equation $y' + y\tan(x) - sin(x) = 0$

② a) Solve the following differential equations:

$$y'' + y = xe^x, \qquad y'' + y = \sin(x) + 2\cos(x)$$

b) Deduce the solutions of the equation $y'' + y = xe^x + \sin(x) + 2\cos(x)$

Exercise 09 : (*)

We consider the following second-order differential equation

$$y'' - 4y' + 4y = 2ch(2x)...(E)$$

① Solve the homogeneous equation associated with (E)

② Give the particular solution of the equation:

$$y'' - 4y' + 4y = e^{2x} ...(E_1)$$

③ Give the particular solution of the equation:

$$y'' - 4y' + 4y = e^{-2x} ...(E_2)$$

④ Deduce the particular solution of the equation (E)

⑤ Give the general solution of the equation (E)

Exercise 10 : (Exam 2013 University of A.Mira-Béjaia) (*)

① We consider the first-order differential equation

$$y' + \left(1 - \frac{1}{x}\right) y = x \qquad \text{(E)}$$

a) Resolve the homogeneous equation associated with (E)

b) Check that $y_p(x) = x$ is a particular solution of (E).

c) Deduce the general solution of the equation (E).

d) Calculate the solution y_1 of (E) satisfying $y_1(1) = 1 + \frac{1}{e}$

① Solve the following differential equation:

$$y'' + y = x^3 + 1$$

Exercise 11 : (Exam 2015-2016 University of A.Mira-Béjaia)

We consider the second-order differential equation

$$y'' - 2y' = 12x - 10...(E) \qquad \text{(E)}$$

1) Solve the associated homogeneous equation for (E)

2) Find the constants a and b such that $y_p(x) = ax^2 + bx$ is a particular solution of (E).

3) Find a solution of (E) satisfying $y(0) = 1$ and $y'(0) = 4$.

Exercise 12 : (Exam 2012-2013 University of A.Mira-Béjaia)

1) Resolve the following differential equations.:

a) $y' + y = e^x$

b) $y' + y^2 = 0$

2) Resolve the following differential equations. :

$$y'' - y' - 6y = \cos(x) + x^2...(E) \qquad \text{(E)}$$

Exercise 13 : (Exam 2017-2018 University of A.Mira-Béjaia)

We consider the following second-order differential equation.

$$y'' - 2y' + y = (6x+2)e^x ...(E)$$

① Solve the associated homogeneous equation for (E)

② Find the constants a and b such that $y_p(x) = (ax^3 + bx^2)e^x$ is a particular solution of (E)."

④ Determine the general solution of (E).

③ Find a solution of (E) satisfying $y(0) = 1$ and $y'(0) = 2$.

Exercise 14 : (Exam 2017-2018 University of A.Mira-Béjaia)

We consider the following second-order differential equation

$$y'' - 5y' + 4y = e^x + 2xe^{4x} ...(E)$$

① Resolve the associated homogeneous equation for (E)

② Determine the general solution of" (E).

③ Find a solution of (E) satisfying $y(0) = 1$ and $y'(0) = 2$.

Exercise 15 : (Exam 2023 University of M'sila)

A) (**2pts**) Resolve the following differential equation:

$$xy' - y = x$$

,

B) We consider the following differential equations.

$$y'' - 5y' + 6y = \frac{1}{2}e^{3x}...(1), \quad y'' - 5y' + 6y = -\frac{1}{2}e^{-3x}...(2), \quad y'' - 5y' + 6y = sh(3x)...(E)$$

④ (**4pts**) Solve the equations (1) and (2), and then deduce the general solution of (E)..