4 – ELECTROSTATIC POTENTIAL ENERGY AND ELECTRIC POTENTIAL

Electric potential energy is the energy associated with the configuration of charges in an electric field. It represents the work done to assemble the system of charges or the work that can be done by the electric field when the configuration of charges changes.

Electric potential energy is a scalar quantity and is measured in joules (J) in the International System of Units (SI). It plays a crucial role in understanding the behavior of charged particles in electric fields and is essential in various fields such as electromagnetism, electronics, and electrochemistry

Electric potential, also known as electric potential energy per unit charge, is a fundamental concept in physics that describes the electric potential energy associated with a point in space due to an electric field.

Imagine you have a positively charged object, like a proton, in space. This charged object creates an electric field around it. Electric potential at any point in space around this charged object is the amount of electric potential energy that a unit positive charge would have if placed at that point.

Mathematically, electric potential at a point in space is defined as the work done per unit charge in bringing a positive test charge from infinity to that point, against the electric field

Electric potential is a scalar quantity, meaning it only has magnitude and no direction. However, it provides a useful way to understand the behavior of electric fields and the interaction between charges in electrostatic systems. In essence, it provides a measure of how much electric potential energy a charge would possess if it were placed at a certain point in space relative to a reference point, often taken to be infinity.

The unit of electric potential in the International System of Units (SI) is volts (V), which is equivalent to one joule per coulomb (J/C).

4 -1 ELECTRIC POTENTIAL ENERGY

4 -1–1 POTENTIAL OF TWO POINT CHARGE

When an external force does work in taking a body from a point to another against a force like spring force or gravitational force, that work gets stored as potential energy of the body. When the external force is removed, the body moves, gaining kinetic energy and losing an equal amount of potential energy. The sum of kinetic and potential energies is thus conserved. Forces of this kind are called conservative forces. Spring force and gravitational force are examples of conservative force

Coulomb force between two (stationary) charges is also a conservative force. This is not surprising, since both have inverse-square dependence on distance and differ mainly in the proportionality constants – the masses in the gravitational law are replaced by charges in Coulomb's law. Thus, like the potential energy of a mass in a gravitational field, we can define electrostatic potential energy of a charge in an electrostatic field

<u>*Conservative electric field and electric force*</u>$\vec{\boldsymbol{E}}$

Let the force vector going from the point '$\boldsymbol{a}$' to the point 'b' through the path I. We try to calculate the line integral from this vector along that path.

$$\int_{Path\ I} \vec{F} \circ d\vec{l} = \int_a^b \vec{F} \circ d\vec{l}$$

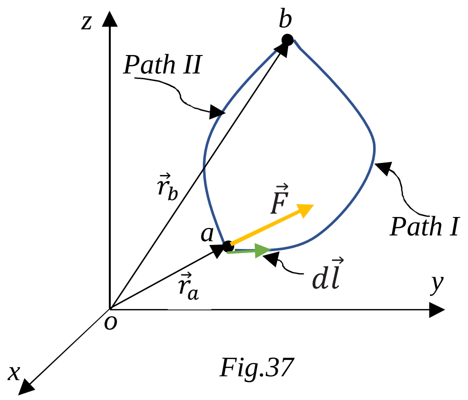


Fig.37

The force is inversely proportional to the squared distance from the origin, then the force can be expressed as:

$$\vec{\boldsymbol{F}} = \frac{\boldsymbol{\alpha}}{\boldsymbol{r}^2}\vec{\boldsymbol{u}}_{\boldsymbol{r}}$$

$$\int_a^b \vec{F} \circ d\vec{l} = \int_a^b \frac{\alpha}{r^2}\vec{u}_r \circ d\vec{l}$$

But $$\boldsymbol{d\vec{l} = dr\,\vec{u}_r + rd\theta\,\vec{u}_\theta + r\sin\theta\,d\varphi\,\vec{u}_\varphi}$$

$$\Rightarrow \quad \frac{\boldsymbol{\alpha}}{\boldsymbol{r}^2}\vec{\boldsymbol{u}}_{\boldsymbol{r}} \circ \boldsymbol{d\vec{l}} = \frac{\boldsymbol{\alpha}}{\boldsymbol{r}^2}\,\boldsymbol{dr}$$

$$\Rightarrow \quad \int_a^b \frac{\boldsymbol{\alpha}}{\boldsymbol{r}^2}\,\boldsymbol{dr} = \boldsymbol{\alpha}\left(\frac{\boldsymbol{1}}{\boldsymbol{r_a}} - \frac{\boldsymbol{1}}{\boldsymbol{r_b}}\right)$$

We see from this that the integral is same whether the path we take (Path I or Path II). So, the integral is independent on the path.

$$\int_{Path\ I} \vec{F} \circ d\vec{l} = \int_{Path\ II} \vec{F} \circ d\vec{l} = \boldsymbol{\alpha}\left(\frac{\boldsymbol{1}}{\boldsymbol{r_a}} - \frac{\boldsymbol{1}}{\boldsymbol{r_b}}\right)$$

If we start from point ' $\boldsymbol{a}$ ' and return to the same point ($\boldsymbol{a} \equiv \boldsymbol{b}$), we trace a loop, and the integral along the loop be zero

$$\oint \vec{F} \circ d\vec{l} = 0$$

The vector who has this property is said to be conservative

In the same manner we found that the electrostatic field is conservative

$$\int_{Path\ I} \vec{E} \circ d\vec{l} = \int_{Path\ II} \vec{E} \circ d\vec{l} = \boldsymbol{\beta}\left(\frac{\mathbf{1}}{\boldsymbol{r_a}} - \frac{\mathbf{1}}{\boldsymbol{r_b}}\right)$$

Where $\vec{\boldsymbol{E}} = \frac{\boldsymbol{\beta}}{\boldsymbol{r^2}}\vec{\boldsymbol{u}}_{\boldsymbol{r}}$

Let $'Q'$ be a source charge which produces an electrostatic field everywhere in the space, and $'q_0'$a positive charge displaced in the field created by the source charge. To displace the charge $'q_0'$ from the point $'\boldsymbol{a}'$ at position $'\boldsymbol{r_a}'$to point $'\boldsymbol{b}'$ at position $\boldsymbol{r_b}$.

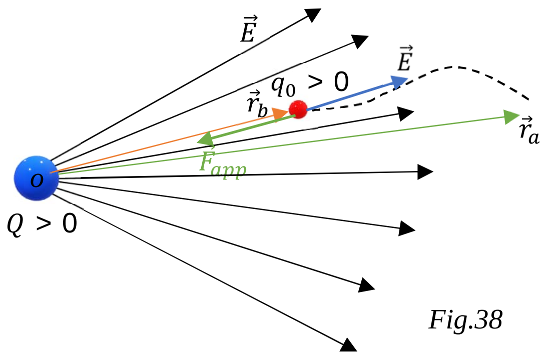


Fig.38

The work done by an exterior agent to bring the charge $'q_0'$ from the position$'\boldsymbol{a}'$ to the position $'\boldsymbol{b}'$is given by:

$$\boldsymbol{W_{ab}} = \int_a^b \vec{\boldsymbol{F}}_{\boldsymbol{app}} \circ \boldsymbol{d\vec{l}}$$

If the displacement is done at constant velocity, then the applied force by the exterior agent, to bring the charge from $\boldsymbol{a}$ is $\boldsymbol{b}$, is opposite to electrostatics force

$$\boldsymbol{W_{ab}} = \int_a^b \frac{\mathbf{1}}{\boldsymbol{4\pi\epsilon_0}} \frac{\boldsymbol{Qq_0}}{\boldsymbol{r^2}} \circ \boldsymbol{d\vec{l}} = \int_a^b \frac{\boldsymbol{Qq_0}}{\boldsymbol{4\pi\epsilon_0}} \frac{\boldsymbol{d\vec{r}}}{\boldsymbol{r^2}} = \frac{\boldsymbol{Qq_0}}{\boldsymbol{4\pi\epsilon_0}}\left(\frac{\mathbf{1}}{\boldsymbol{r_b}} - \frac{\mathbf{1}}{\boldsymbol{r_a}}\right)$$

The electrical potential energy is equal to the work done by an exterior agent to bring the charge from a to b

$$\boldsymbol{U_f} - \boldsymbol{U_i} = \boldsymbol{U_b} - \boldsymbol{U_a} = \boldsymbol{W_{ab}} = \frac{\boldsymbol{Qq_0}}{\boldsymbol{4\pi\epsilon_0}}\left(\frac{\mathbf{1}}{\boldsymbol{r_b}} - \frac{\mathbf{1}}{\boldsymbol{r_a}}\right)$$

We know that the potential energy is computed from certain reference. If we take that reference as infinity. So, bringing the charge from infinity to any point with position r, gives:

$$\boldsymbol{U(r)} - \boldsymbol{U_\infty} = \boldsymbol{U(r)} = \frac{\boldsymbol{Qq_0}}{\boldsymbol{4\pi\epsilon_0}}\frac{\mathbf{1}}{\boldsymbol{r}}$$

4 -2 –2 POTENTIAL ENERGY FOR ASSEMBLY OF POINT CHARGES

Let our system constituted by n charges q_1, q_2, ..., q_n. What is the amount of potential energy to constitute a specific configuration with this system of charges?

We begin in putting a charge q_1 at any position. The second charge q_2 is bringing at a distance $\vec{r}_{12}$ from the charge q_1. Then the potential energy for this system of two point charges is:

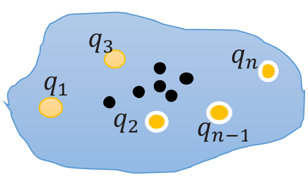


Fig.39

$$U_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}$$

In bringing the third charge q_2 at the vicinity of system of the two point charges q_1 and q_2. The charge interacts with this two charges and gives the potential energy:

$$U_3 = U_{13} + U_{23}$$

$$U_{13} = \frac{q_1 q_3}{4\pi\epsilon_0 r_{13}}; \qquad U_{23} = \frac{q_2 q_3}{4\pi\epsilon_0 r_{23}}$$

If we assemble all the point charges in certain configuration, we get the total potential energy necessary to constitute this configuration

$$U = U_{12} + U_{13} + U_{23} + \ldots + U_{1n} + U_{2n} + \ldots + U_{(n-1)n}$$

$$U = \frac{1}{2} \sum_{(i,j)=1}^{n} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$$

4 -2 ELECTRIC POTENTIAL

The electric potential is the work done to bring a unit charge from infinity to a position r. Or the work done per unit charge.

$$V(r) = \int_{ref}^{r} \vec{E} \circ d\vec{l}$$

4 -2 –1 POTENTIAL DUE TO POINT CHARGE

The potential energy is given by

$$U(r) = \frac{Q q_0}{4\pi\epsilon_0} \frac{1}{r}$$

Then the work done by unit charge is

$$\frac{U(r)}{q_0} = \frac{Q}{4\pi\epsilon_0 r} = V(r)$$

The potential depends only on the charge Q itself and the point considered in the space

4 -2 –2 POTENTIAL DUE TO SEVERAL POINT CHARGE

Now we treat the case when we have multiple charge. What is the electric potential created by this system $q_1, q_2, \ldots, q_n$?

The charge q_1 produces a potential V_1 at point P:

$$V_1 = \frac{1}{4\pi\epsilon_0 r_1}$$

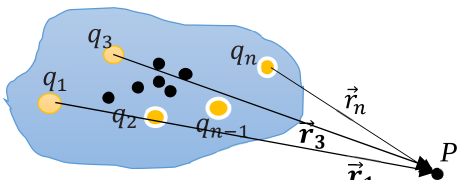


Fig.40

The charge q_2 produces a potential V_2 at point P:

$$V_2 = \frac{1}{4\pi\epsilon_0 r_2}$$

Until the charge q_n which produces a potential V_n at point P:

$$V_n = \frac{1}{4\pi\epsilon_0 r_n}$$

The total electric potential is the scalar sum of the all potential that created par each charge

$$V = \sum_{i=1}^{n} V_i = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$$

4 -2 –3 ELECTRIC POTENTIAL DUE TO CONTINUOUS DISTRIBUTION OF CHARGE

The electric potential at a point A in an electric field is defined as the external work done in bringing slowly a unit positive test charge against the electric field from infinity to that point A.

A – LINE DISTRIBUTION WITH LINEAR CHARGE DENSITY

Let a charge which has a charge distributed along a line with linear charge density $\boldsymbol{\lambda}$. *We try to found the electric potential created by this distribution in any point of the space. For this we use a thin rod of length* $\boldsymbol{L}$ *with a uniform linear distribution* λ

To calculate the electric potential in the point P due the charge of the rod, we take a very small piece of rod which has the dimension dy. This element of length contains a charge $\boldsymbol{dq = \lambda\, dy}$.

That element, treated as point charge produces an infinitesimal electric potential $\boldsymbol{dV}$ *given by:*

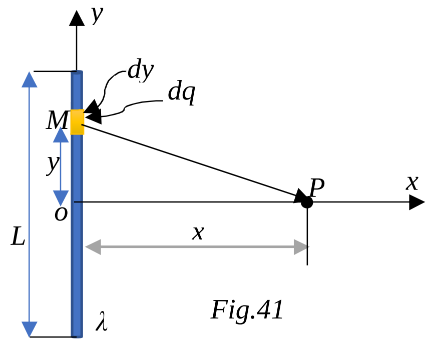


Fig.41

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} = \frac{\lambda}{4\pi\epsilon_0 r}dy$$

The total potential is equal to the sum of all effect of the whole charge

$$V(P) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\lambda}{4\pi\epsilon_0 r}\,dy$$

But the distance $\boldsymbol{r}$ *is given by:*

$$r = |\overrightarrow{\mathbf{MP}}| = \sqrt{x^2 + y^2}$$

$$V(P) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\lambda}{4\pi\epsilon_0}\frac{dy}{\sqrt{x^2+y^2}} = \frac{\lambda}{4\pi\epsilon_0}\left[Ln\left(y + \sqrt{x^2+y^2}\right)\right]_{-L/2}^{L/2}$$

$$V(P) = \frac{\lambda}{4\pi\epsilon_0}\left[Ln\left(\frac{L}{2} + \sqrt{x^2 + \frac{L^2}{4}}\right) - Ln\left(-\frac{L}{2} + \sqrt{x^2 + \frac{L^2}{4}}\right)\right]$$

B – SURFACE DISTRIBUTION WITH SURFACE CHARGE DENSITY

Now, we calculate the electric potential due to a charge distributed on surface with a charge density σ

To calculate the electric potential in the point P due the charge of the disc. We take a very small piece of disc which has the dimension ds. This element of length contains a charge

$\boldsymbol{dq = \sigma\, ds}$

That element, taken as point charge produces an infinitesimal electric potential $\boldsymbol{dV}$ given by:

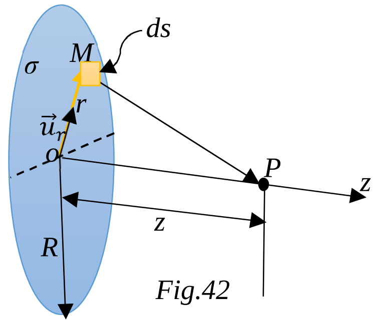


Fig.42

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} = \frac{\sigma}{4\pi\epsilon_0|\overrightarrow{MP}|}ds$$

But the distance $|\overrightarrow{MP}|$ is given by: $\overrightarrow{MP} = \overrightarrow{MO} + \overrightarrow{OP} = -r\,\vec{u}_r + z\,\vec{k}$

$$|\overrightarrow{MP}| = \sqrt{r^2 + z^2}$$

The element of surface is: $\boldsymbol{ds = r\,dr\,d\theta}$

The total potential is equal to the sum of all effect of the whole charge

$$V(P) = \int_0^{2\pi}\int_0^R \frac{\sigma}{4\pi\epsilon_0}\frac{r}{\sqrt{r^2+z^2}}\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^R \frac{\sigma}{4\pi\epsilon_0}\frac{r}{\sqrt{r^2+z^2}}\,dr$$

$$V(P) = \frac{\sigma}{2\epsilon_0}\sqrt{r^2+z^2}\Big|_0^R = \frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2+z^2} - |z|\right)$$

$$V(P) = \frac{\sigma}{2\epsilon_0}\begin{cases}\sqrt{R^2+z^2} + z & if\ z < 0\\ \sqrt{R^2+z^2} - z & if\ z > 0\end{cases}$$

C – VOLUME DISTRIBUTION WITH VOLUMS CHARGE DENSI

If the charge is distributed in the volume with a charge density $\boldsymbol{\rho}$. To calculate the electric potential at point P in the space. To do this we proceed as follows:

- *Take an element of volume dv*
- *Assign the infinitesimal charge $\boldsymbol{dq}$ to that volume element*
- *Assume that element of charge produces an electric potential at point P*
- *Add all element of electric potential. This is done by integration over all charged volume*

Calculate the electric field of a uniformly charged sphere on volume with density of charge $\boldsymbol{\rho}$ at point $\boldsymbol{P}$ exterior to the sphere.

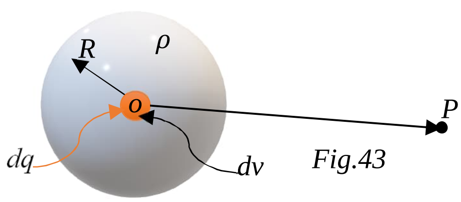


Fig.43

Let the volume element $dv = 4\pi r^2 dr$ whose charge is dq.

That charge is equal to: $dq = \rho\, dv$

This charge, produces an electric potential dV *given by:*

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{|\overrightarrow{OP}|} = \frac{\rho}{\epsilon_0}\frac{dv}{|\overrightarrow{OP}|} = \frac{\rho}{\epsilon_0}\frac{r^2}{|\overrightarrow{OP}|}dr$$

To compute the total electric potential due to the whole charge we integrate the expression above over the sphere with r *varies between* 0 *and* R *in taking in count that the reference is infinity and the potential at that point is zero*

$$V = \int_0^R \frac{\rho}{\epsilon_0}\frac{r^2}{|\overrightarrow{OP}|}dr = \frac{\rho}{3\epsilon_0}\frac{R^3}{r}$$

$$|\overrightarrow{OP}| = r$$

In general, to calculate a potential due to a continuous distribution, we take an element of charge assumed to be point charge. Then find the expression of potential for that element of charge. Finally, to compute the effect of the whole charge we integrate over the shape of the distributed charge

$$dq \quad \rightarrow \quad \begin{cases} \lambda\, dl \\ \sigma\, ds \\ \rho\, dv \end{cases} \quad \rightarrow \quad dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} \quad \rightarrow \quad V = \begin{cases} \int \frac{\lambda}{4\pi\epsilon_0}\frac{dl}{r} \\ \iint \frac{\sigma}{4\pi\epsilon_0}\frac{ds}{r} \\ \iiint \frac{\rho}{4\pi\epsilon_0}\frac{dv}{r} \end{cases}$$

4– 3 GRAPHICAL REPRESENTATIONS OF ELCTRIC POTENTIAL: EQUIPOTENTIEL SURFACES

Like a field line, we can represent the electric potential in different point of the space. The graphics are not a curve but a surface. In such surface it represents the same potential over all its point which we call equipotential surface

An equipotential surface is a surface on which the electric potential is the same everywhere. The easiest equipotential surfaces to visualize are those that surround an isolated point charge. The potential at a distance r *from a point charge* q *is* $V = kq/r$*. Thus, wherever* r *is the same, the potential is the same, and the equipotential surfaces are spherical surfaces centered on the charge. There are an infinite number of such surfaces, one for every value of r. The larger the distance r, the smaller is the potential of the equipotential surface.*

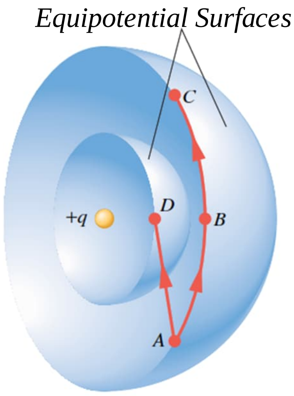


Fig.44

Notice that the equipotential surfaces are represented in the same manner for positive point charge as well as for a negative point charge.

For two equal opposite charges, the equipotential surfaces are a deformed circle. They are squeezed between the charges because the field in that region is intense.

The field lines are always orthogonal to the equipotential surfaces

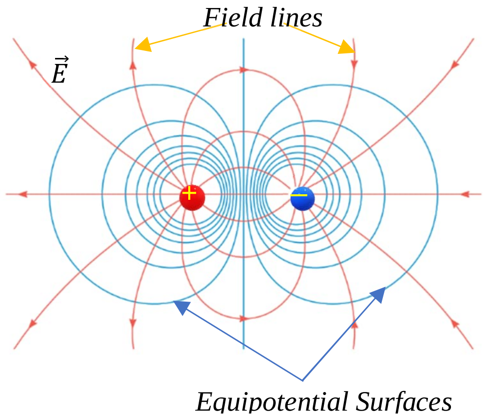


Fig.45

Since the field lines are always orthogonal to the equipotential surfaces, so, we can find the equation of these surfaces by resolving the following differential equation $\vec{E} \circ d\vec{l} = 0$.

$$\vec{\mathbf{E}} \circ \boldsymbol{d\vec{l}} = \boldsymbol{E_x dx} + \boldsymbol{E_y dx} + \boldsymbol{E_z dz} = \mathbf{0}$$

4 – 4 RELATIONSHIP BETWEEN ELECTRIC FIELD AND ELECTRIC POTENTIAL

We calculate the curl of the electric field and it was zero. This is due to the conservative nature.

The vector field for a point charge is: $\boldsymbol{\vec{E}} = \frac{\mathbf{1}}{\boldsymbol{4\pi\varepsilon_0}}\frac{\boldsymbol{Q}}{\boldsymbol{r^2}}\boldsymbol{\vec{u}_r} = \frac{\boldsymbol{\alpha}}{\boldsymbol{r^2}}\boldsymbol{\vec{u}_r}$

$$\vec{\boldsymbol{\nabla}} \wedge \boldsymbol{\vec{E}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \mathbf{0}$$

The vector electrostatics field is irrotational. That property characterizes the conservative quantity. So, the electric field is conservative

Another way to express this feature is that its line integral between two points A *and* B *or its integral along a loop is null*

$$\oint \vec{E} \,.\, d\vec{l} = 0$$

Using this fact and the two theorems, gradient theorem and curl (STOKES) theorem.

$$\int_{Path} \vec{\nabla} f \circ d\vec{l} = \int_A^B df = F(B) - F(A)$$

$$\oint \vec{E} \,.\, d\vec{l} = 0 = \iint (\vec{\nabla} \wedge \boldsymbol{\vec{E}}) \circ d\vec{s}$$

$$\vec{\nabla} \wedge \vec{E} = \mathbf{0}$$

There is another identity in vector calculus: $\vec{\nabla} \wedge \vec{\nabla} f = 0$

$$\int_{Path} \vec{\nabla} V \circ d\vec{l} = \int dV = \Delta V = -\oint \vec{E} \circ d\vec{l} = -\iint (\vec{\nabla} \wedge \vec{E}) \circ d\vec{s} = \int_{Path} \left(\vec{\nabla} \wedge (\vec{\nabla} f)\right) \circ d\vec{s}$$

$$\vec{E} = -\vec{\nabla} V$$

5 – ELECTRIC DIPOLE

A pair of equal and opposite charges separated by a small distance is called an electric dipole. The dipole is characterized by its dipole moment which is a vector whose magnitude is either charge times the separation between the two opposite charges and the direction is along the dipole axis from the negative to the positive charge.

The dipole moment is given by: $\vec{P} = Q.\overrightarrow{AB} = Q.\vec{a}$

5 – 1 ELECTRICAL POTENTIAL AND ELECTRICAL FIELD FOR A DIPOLE

ELECTRICAL POTENTIAL

At point will be created a potential equal to the sum of the potential created by each point charge

$$V(M) = V_A + V_B$$

$$V_A = \frac{1}{4\pi\epsilon_0} \frac{-Q}{r_-} \quad , \quad V_B = \frac{1}{4\pi\epsilon_0} \frac{Q}{r_+}$$

$$V(M) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

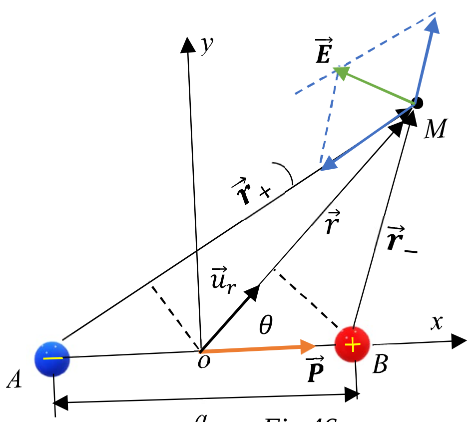


Fig.46

$\vec{r}_+ = \overrightarrow{AO} + \overrightarrow{OM}$ *and* $\vec{r}_- = \overrightarrow{BO} + \overrightarrow{OM}$

But

$\overrightarrow{AM} = \frac{a}{2}\,\vec{\imath} + \vec{r}$ *and* $\overrightarrow{BM} = \vec{r}_- = -\frac{a}{2}\vec{\imath} + \vec{r}$ *with* $\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath}$

Then

$$\overrightarrow{AM} = \left(x + \frac{a}{2}\right)\vec{\imath} + y\,\vec{\jmath} \qquad \overrightarrow{BM} = \left(x + \frac{a}{2}\right)\vec{\imath} + y\,\vec{\jmath}$$

$$\left|\overrightarrow{AM}\right| = r_+ = \sqrt{\left(x + \frac{a}{2}\right)^2 + y^2} \qquad \left|\overrightarrow{BM}\right| = r_- = \sqrt{\left(x - \frac{a}{2}\right)^2 + y^2} \qquad r = \sqrt{x^2 + y^2}$$

$$V(M) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{\left(x+\frac{a}{2}\right)^2 + y^2}} - \frac{1}{\sqrt{\left(x-\frac{a}{2}\right)^2 + y^2}}\right)$$

Which is a general expression of the potential due to two opposite point charges. To calculate the potential for a dipole the point M is very far away ($r >> \frac{a}{2}$)

In this condition, we have an approximation to do

$$\sqrt{\left(x+\frac{a}{2}\right)^2 + y^2} = \sqrt{(x^2+y^2) + ax + \left(\frac{a}{2}\right)^2} = r\sqrt{1 + \frac{ax}{r} + \left(\frac{a}{2r}\right)^2}$$

Neglecting the term $\left(\frac{a}{2r}\right)^2$ for this approximation and use the limited expansion, we get

$$\frac{1}{r_+} = \frac{1}{r}\left(1+\frac{ax}{r}\right)^{-1/2} = \frac{1}{r}\left(1+\frac{ax}{r^2}\right)^{-1/2} = \frac{1}{r}\left(1-\frac{1}{2}\frac{ax}{r^2}\right)$$

$$\frac{1}{r_-} = \frac{1}{r}\left(1-\frac{ax}{r}\right)^{-1/2} = \frac{1}{r}\left(1-\frac{ax}{r^2}\right)^{-1/2} = \frac{1}{r}\left(1+\frac{1}{2}\frac{ax}{r^2}\right)$$

$$V(M) = \frac{Q}{4\pi\epsilon_0}\frac{ax}{r^3} = \frac{Q}{4\pi\epsilon_0}\frac{ax}{(x^2+y^2)^{3/2}}$$

Which can be written in the form

$$V(M) = \frac{1}{4\pi\epsilon_0}\frac{Qax}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{(Qa)(r\,cos\theta)}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{\vec{P}\circ\vec{u}_r}{r^2}$$

The potential depends on the inverse of squared distance between midpoint of the dipole and the point M.

ELECTRICAL FIELD

The electric field is calculated by using the relationship between the potential and the electric field $\vec{E} = -\vec{\nabla}V$

We can use the cylindrical coordinate system or cartesian coordinate system

- *Cylindrical coordinate system*

$$\vec{E} = E_r\vec{u}_r + E_\theta\vec{u}_\theta + E_z\vec{k} = -\vec{\nabla}V = -\left(\frac{\partial V}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\vec{u}_\theta + \frac{\partial V}{\partial z}\vec{k}\right)$$

The radial component

$$E_r = -\frac{\partial V}{\partial r} = -\frac{1}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{P\,cos\theta}{r^2}\right) = \frac{1}{4\pi\epsilon_0}\frac{2\,P\,cos\theta}{r^2}$$

The transverse component

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\frac{1}{4\pi\epsilon_0}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{P\cos\theta}{r^2}\right) = \frac{1}{4\pi\epsilon_0}\frac{P\sin\theta}{r^2}$$

$$E_z = 0$$

The electric field has the components $\begin{cases} E_r = \frac{1}{4\pi\epsilon_0}\frac{2P\cos\theta}{r^2} \\ E_\theta = \frac{1}{4\pi\epsilon_0}\frac{P\sin\theta}{r^2} \\ E_z = 0 \end{cases}$

- *Cartesian coordinate system*

$$\vec{E} = E_x\,\vec{i} + E_y\vec{j} + E_z\vec{k} = -\vec{\nabla}V = -\left(\frac{\partial V}{\partial x}\vec{i} + \frac{\partial V}{\partial y}\vec{j} + \frac{\partial V}{\partial z}\vec{k}\right)$$

The 'x' component

$$E_x = -\frac{\partial V}{\partial x} = -\frac{1}{4\pi\epsilon_0}\frac{\partial}{\partial x}\left(\frac{Px}{(x^2+y^2)^{3/2}}\right) = \frac{P}{4\pi\epsilon_0}\left(\frac{1}{(x^2+y^2)^{3/2}} - \frac{3x^2}{(x^2+y^2)^{5/2}}\right)$$

The 'y' component

$$E_y = -\frac{\partial V}{\partial x} = -\frac{1}{4\pi\epsilon_0}\frac{\partial}{\partial x}\left(\frac{Px}{(x^2+y^2)^{3/2}}\right) = \frac{P}{4\pi\epsilon_0}\left(\frac{3xy}{(x^2+y^2)^{5/2}}\right)$$

The 'z' component

$$E_z = -\frac{\partial V}{\partial z} = 0$$

The electric field has the components $\begin{cases} E_x = \frac{P}{4\pi\epsilon_0}\left(\frac{1}{(x^2+y^2)^{3/2}} - \frac{3x^2}{(x^2+y^2)^{5/2}}\right) \\ E_y = \frac{P}{4\pi\epsilon_0}\left(\frac{3xy}{(x^2+y^2)^{5/2}}\right) \\ E_z = 0 \end{cases}$

5 – 2 A DIPOLE IN UNIFORME ELECTRICAL FIELD

When the dipole is in a uniform electric field $\vec{E} = E_0\,\vec{j}$*, each charge experiences a force*

The positive charge experiences a force $\vec{F}_+ = Q.\vec{E}$ *which has the same direction as the electric field*

The negative charge experiences a force $\vec{F}_- = -Q.\vec{E}$ *which has the opposite direction to the electric field*

We see that the net force is zero

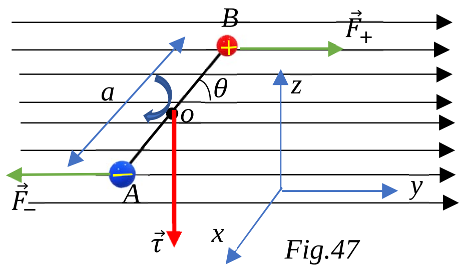


Fig.47

$$\vec{F}_{net} = \vec{F}_+ + \vec{F}_- = \vec{0}$$

The dipole has no translational motion. But the application points of these forces are not same. So, they produce a rotational motion, about the center of mass of the dipole characterized by the torque $\vec{\tau}$

The force $\vec{F}_+$ *Produces a torque* $\vec{\tau}_+$*:* $\vec{\tau}_+ = \overrightarrow{OA} \wedge \vec{F}_+ = \overrightarrow{OA} \wedge (Q.\vec{E})$

The force $\vec{F}_-$ *Produces a torque* $\vec{\tau}_-$*:* $\vec{\tau}_- = \overrightarrow{OB} \wedge \vec{F}_- = \overrightarrow{OB} \wedge (-Q.\vec{E}) = -\overrightarrow{OB} \wedge (Q.\vec{E})$

The net torque that a dipole experience is:

$$\vec{\tau} = \vec{\tau}_+ + \vec{\tau}_- = Q(\overrightarrow{OA} - \overrightarrow{OB}) \wedge \vec{E} = Q\overrightarrow{AB} \wedge \vec{E}$$

$$\vec{\tau} = \vec{P} \wedge \vec{E} = -PE\, sin\theta\, \vec{k}$$

The torque has the tendency to align the dipole along the direction of the field

In the case of a molecule assimilated to a dipole, point A represents the center of gravity of the negative charges and point B the center of gravity of the charges

ELECTRICAL POTENTIAL ENERGY

When the torque rotates the dipole, amount of work is done in the system, which is related to the potential energy

$$\Delta U = U_B - U_A = Q(V_B - V_A)$$

The infinitesimal work done in the system is

$$dW = \tau.d\theta = PE\, sin\theta d\theta$$

$$\Longrightarrow \quad W = \int_{\frac{\pi}{2}}^{\theta} -E\, sin\theta d\theta = -PE\, cos\theta \Big|_{\frac{\pi}{2}}^{\theta} = -PEcos\theta$$

$$U = -PEcos\theta = -\vec{P} \circ \vec{E}$$

5 – 3 A DIPOLE IN NON-UNIFORM ELECTRICAL FIELD

When the dipole is placed in non-uniform electric field $\vec{E}(\vec{r})$*, the forces experienced by the negative and positives are opposites but not with the same magnitudes. So, the net force in this case will not be zero. The dipole seems to be in translational motion, in addition to a rotational motion due to the presence of a torque.*

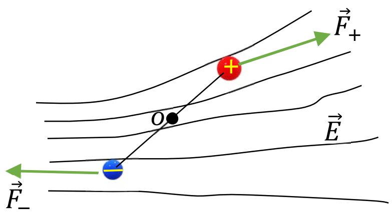


Fig.48

6 – ELECTRIC FLUX AND GAUSS's THEOREM

6 -1 Solid Angle

The plane angle is the surface defined between two lines. If we want to sweep the whole plane, the angle is $\boldsymbol{2\pi}$ *(Fig.49 - a)*

We see the arc (curved line) $\boldsymbol{AB}$ *is subtended by the angle* θ *such that its length is* $\boldsymbol{AB = R\theta}$

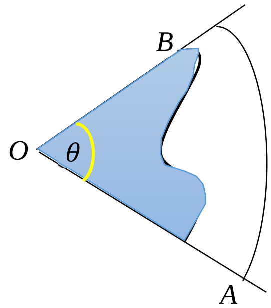


Fig.49-a

If we want to see the surface $\boldsymbol{S}$ *subtended by a cone from the point O we will sweep an angle in the space, such angle is called solid angle. The unit is* <u>*steradian*</u>

The solid angle Ω *is the angle under which we see the surface* $\boldsymbol{S}$ *at the distance* $\boldsymbol{R}$ *from the point* $\boldsymbol{O}$ *in all space.*

$$\Omega = \frac{\boldsymbol{S}}{\boldsymbol{R^2}}$$

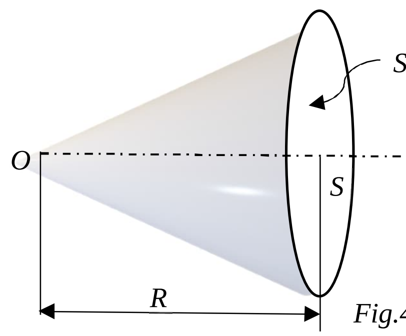


Fig.49-B

For an element of surface on the sphere

$$\boldsymbol{dS = r^2 \sin\theta \, d\theta \, d\varphi} \quad \Longrightarrow \quad \boldsymbol{d\Omega = \frac{dS}{r^2}}$$

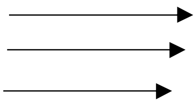
$$\boldsymbol{d\Omega = \sin\theta \, d\theta \, d\varphi}$$

The total angle subtended by all space is:

$$\Omega = \int \boldsymbol{d\Omega} = \iint \boldsymbol{\sin\theta \, d\theta \, d\varphi} = \int_0^{\pi} \boldsymbol{\sin\theta \, d\theta} \int_0^{2\pi} \boldsymbol{d\varphi}$$

$$\boldsymbol{\Omega = 4\pi}$$

6 -2 Electric Flux



The flux is the flow of a vector through a cross section, or the number of force lines of vector that pass normally through that surface (Density of force lines)

Fig.1-a

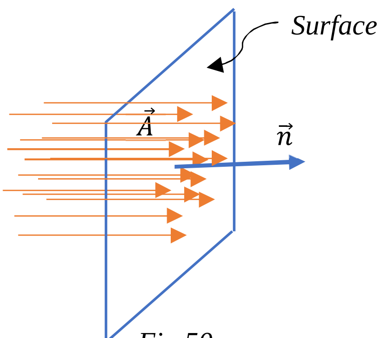


Fig.50-a

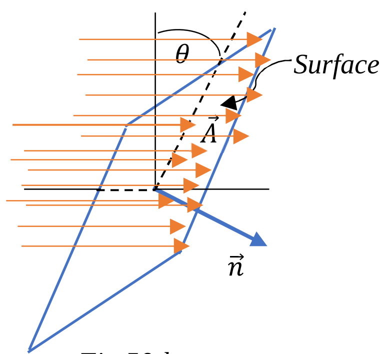


Fig.50-b

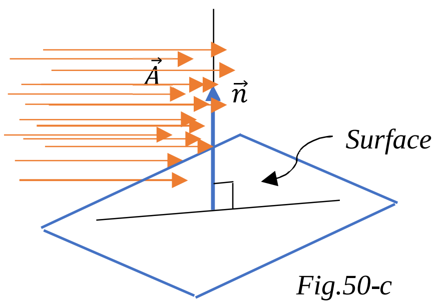


Fig.50-c

From the figures above (Fig. 50 – a, b, c), one can see that in figure left there is a maximum of force lines of the vector $\vec{A}$ that pass through the surface $\boldsymbol{S}$, which is perpendicular to the vector field $\vec{A}$ (the two vectors are parallel $\vec{A} \parallel \vec{S}$). When the surface is inclined with an angle θ the number of force lines decreases, which become zero when the surface is parallel to the direction of this force lines.

From that we can see that the flux is maximal when the surface is orthogonal to the vector field and decrease when this surface is tilted until become null when the surface become parallel to the vector field. So, the flux is proportional to the number of force lines (intensity) of the vector field and the surface penetrating by these lines.

We can express the flux as follows:

$$\Phi = \boldsymbol{A_{\perp} S}$$

$A_{\perp}$: *is the component of the vector that is orthogonal to the surface* $\boldsymbol{A_{\perp}} = \boldsymbol{A\, cos\theta}$

S: *is the area of the surface* $\vec{\boldsymbol{S}} = \boldsymbol{S}\,\vec{\boldsymbol{n}}$

The flux Φ *is a scalar quantity. We can write the expression in the vector form as:*

$$\Phi = \boldsymbol{A_{\perp} S} = \boldsymbol{A}\,.\boldsymbol{cos\theta}.\boldsymbol{S} = \vec{\boldsymbol{A}} \circ \vec{\boldsymbol{S}}$$

If the vector $\vec{A}$ *is an electric field vector, its flux over a surface* $\vec{S}$ *is given by:*

$$\Phi = \vec{\boldsymbol{E}} \circ \vec{\boldsymbol{S}}$$

The unit of the flux is a WEBER which is $\frac{N}{C}\, m^2$

This expression is given for a uniform vector field. We know in general the field is variable for one point to another, So, we can define an element of surface $\boldsymbol{d\vec{S}}$*, through which pass a vector field* $\vec{\boldsymbol{E}}$*, considered constant over that element of surface. And the element that correspond to that is:*

$$\boldsymbol{d}\Phi = \vec{\boldsymbol{E}} \circ \boldsymbol{d\vec{S}}$$

To find the total electrical flux, we must integrate over a whole surface

$$\Phi = \int \boldsymbol{d}\Phi = \iint_{\boldsymbol{s}} \vec{\boldsymbol{E}} \circ \boldsymbol{d\vec{S}}$$

Electrical flux through a circular surface (Disc)

A point charge $\boldsymbol{Q}$ *placed at O produces an electric field* $\vec{\boldsymbol{E}}$*. What is the electric flux through the circular surface of radius* $\boldsymbol{R}$ *at a distance* $\boldsymbol{l}$ *from the point charge?*

The point charge creates an electric field in all direction in the space. The line field that passe through the disc of radius R form a cone, and the electric field at different point of

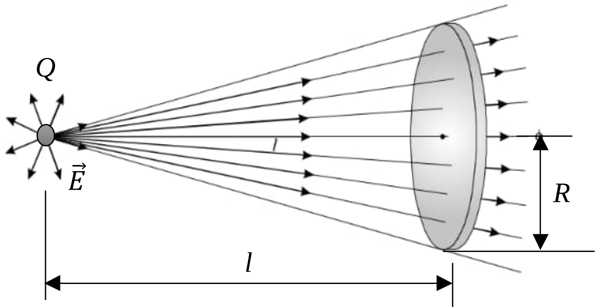


Fig.51-a

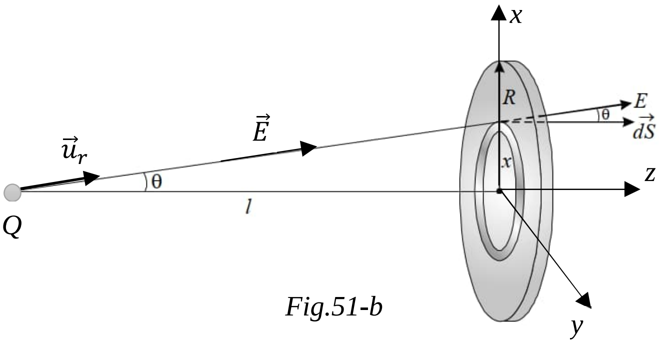


Fig.51-b

its surface is different. So, we use the form integral to calculate the electric flux through that surface.

We take an element of surface $\boldsymbol{d\vec{S}}$ *which cut a portion of the cone and the flus trough that surface is* $\boldsymbol{d}\Phi$

$$\boldsymbol{d}\Phi = \vec{\boldsymbol{E}} \circ \boldsymbol{d\vec{S}} \quad \Rightarrow \quad \Phi = \iint \vec{\boldsymbol{E}} \circ \boldsymbol{d\vec{S}}$$

From the figure we see that

$$\boldsymbol{d\vec{S}} = \boldsymbol{2\pi\, r\, dr\vec{k}}$$

and the electric field on the ring is given by

$$\vec{\boldsymbol{E}} = \frac{\boldsymbol{1}}{\boldsymbol{4\pi\varepsilon_0}}\frac{\boldsymbol{Q}}{(\boldsymbol{r^2+l^2})}\vec{\boldsymbol{u}}_{\boldsymbol{r}}$$

The total electric flux through the disc is given by

$$\Phi = \iint \vec{\boldsymbol{E}} \circ \boldsymbol{d\vec{S}} = \int_0^R \frac{\boldsymbol{1}}{\boldsymbol{4\pi\varepsilon_0}}\frac{\boldsymbol{Q}}{(\boldsymbol{r^2+l^2})}\vec{\boldsymbol{u}}_{\boldsymbol{r}} \circ \boldsymbol{2\pi\, r\, dr}\,\vec{\boldsymbol{k}}$$

$$\Phi = \frac{\boldsymbol{Q}}{\boldsymbol{4\varepsilon_0}}\int_0^R \frac{\boldsymbol{r\, dr}}{(\boldsymbol{r^2+l^2})}\vec{\boldsymbol{u}}_{\boldsymbol{r}} \circ \vec{\boldsymbol{k}} = \frac{\boldsymbol{Q}}{\boldsymbol{4\varepsilon_0}}\int_0^R \frac{\boldsymbol{2r\, dr}}{(\boldsymbol{r^2+l^2})}\boldsymbol{\cos\theta}$$

$$\boldsymbol{\cos\theta} = \frac{\boldsymbol{l}}{\sqrt{\boldsymbol{r^2+l^2}}}$$

$$\Phi = \frac{\boldsymbol{Q\, l}}{\boldsymbol{4\varepsilon_0}}\int_0^R \frac{\boldsymbol{2r\, dr}}{(\boldsymbol{r^2+l^2})^{3/2}} = \frac{\boldsymbol{Q}}{\boldsymbol{2\varepsilon_0}}\left(\boldsymbol{1} - \frac{\boldsymbol{l}}{\sqrt{\boldsymbol{R^2+l^2}}}\right)$$

Now, let see the electric flux of point charge when it is outside the surface or inside it

- Electric flux of point charge outside a closed the surface

The point charge is outside the surface. The first elementary surface encountered by the electrical field is at distance $\boldsymbol{r_1}$ *from the point charge and in the opposite direction of the electric field* $(\vec{E}, d\vec{S}) = 180° = \pi$ *. The second elementary surface encountered is at the distance* $\boldsymbol{r_2}$ *from the point charge and is in the same direction as the electric field* $(\vec{E}, d\vec{S}) = 0°$*.*

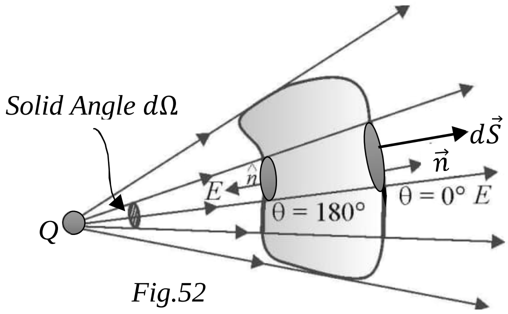


Fig.52

The field created at the first surface is:

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{Q}{r_1^2}\vec{u}_r$$

The flux through that surface is:

$$d\Phi_1 = \vec{E}_1 \circ d\vec{S} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r_1^2}\vec{u}_r \circ d\vec{S} = -\frac{Q}{4\pi\epsilon_0}\frac{dS}{r_1^2} = -\frac{Q}{4\pi\epsilon_0}d\Omega$$

The field created at the first surface is:

$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{Q}{r_2^2}\vec{u}_r$$

The flux through that surface is:

$$d\Phi_2 = \vec{E}_2 \circ d\vec{S} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r_2^2}\vec{u}_r \circ d\vec{S} = \frac{Q}{4\pi\epsilon_0}\frac{dS}{r_2^2} = \frac{Q}{4\pi\epsilon_0}d\Omega$$

The total flux is the sum of all field line through the total surface.

$$d\Phi = d\Phi_1 + d\Phi_2 = 0$$

We see that the number of line field penetrating the surface is the same leaving it so the total flux is zero

- Electric flux of point charge inside a closed the surface

The first elementary surface encountered by the electrical field is at distance $\boldsymbol{r}$ *from the charge. In this case whatever is the element of surface, it is always parallel to the vector field. So, at each point of the surface the flux is positive. The total flux is the sum of this elemets who are positve. From the figure the field lines issue from the charge all leaves the closed surface*

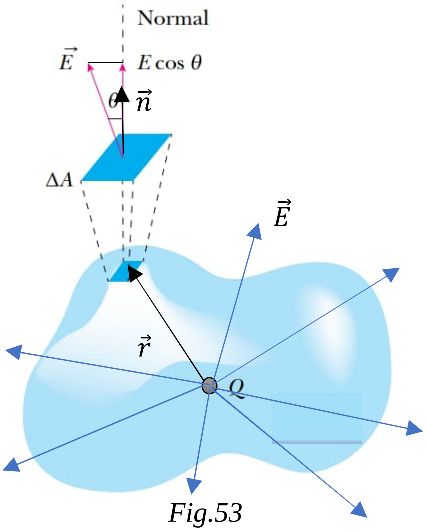


Fig.53

$$d\Phi = \vec{E} \circ d\vec{S} \quad \Rightarrow \quad \Phi = \iint \vec{E} \circ d\vec{S}$$

$$\Phi = \iint \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\vec{u}_r \circ d\vec{S} = \frac{Q}{4\pi\epsilon_0}\iint \frac{dS}{r^2} = \frac{Q}{4\pi\epsilon_0}\iint d\Omega$$

For all space $\mathbf{\Omega} = \mathbf{4\pi}$, then

$$\Phi = \frac{Q}{\epsilon_0}$$

6 -3 GAUSS' Law

In previous calculus, to found electric field due to one., several and continuous density of charge, is a labor intensive when using Coulomb's law. An alternative way to do this in simply few steps, when we exploit the symmetry distribution of the charge. This relationship for finding the field in this manner is called GAUSS 'law

Starting from the fact that the flux is independent on the shape of the closed surface. i.e., all the field lines starting from the point charge pass through the regular surface S', also through the irregular shape S

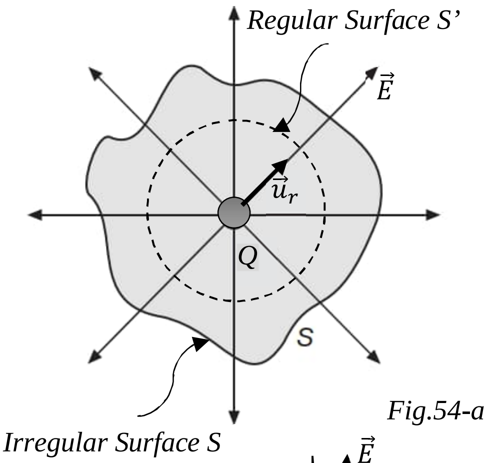


Fig.54-a

$$\Phi = \iint_{S'} \vec{E} \circ d\vec{S} = \iint_{S} \vec{E} \circ d\vec{S}$$

The electric field created by a point charge is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\vec{u}_r$$

The element of surface is:

$$d\vec{S} = dS\,\vec{n}$$

$$\vec{u}_r = \vec{n}$$

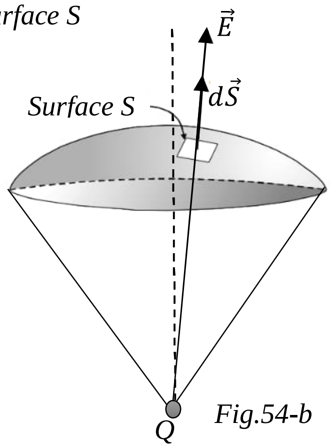


Fig.54-b

$$\Phi = \iint_{S'} \vec{E} \circ d\vec{S} = \iint_{S} E\,dS = \iint_{S} \frac{Q}{4\pi\epsilon_0}\frac{dS}{r^2}\,dS$$

$$\Phi = \iint_{S'} \vec{E} \circ d\vec{S} = \frac{Q}{\epsilon_0}$$

GAUSS' Law

The flux through a surface is equal to the charge enclosed on the permittivity of the medium

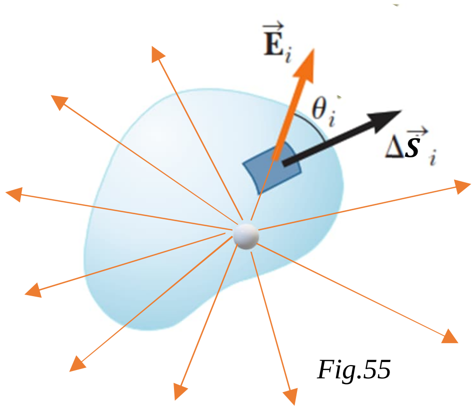


Fig.55

$$\Phi = \iint_{S} \vec{E} \circ d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

Q_{enc} are the all charges inside the GAUSS surface

Some features of Gauss surface

1- *Gauss's law is true for any closed surface, no matter what its shape or size.*
2- *The term $\boldsymbol{Q_{enc}}$ on the right side of Gauss's law, includes the sum of all charges enclosed by the surface. The charges may be located anywhere inside the surface.*
3- *In the situation when the surface is so chosen that there are some charges inside and some outside, the electric field is due to all the charges, both inside and outside S. The term $\boldsymbol{Q_{enc}}$ on the right side of Gauss's law, however, represents only the total charge inside S.*
4- *The surface that we choose for the application of Gauss's law is called the Gaussian surface. You may choose any Gaussian surface and apply Gauss's law. However, take care not to let the Gaussian surface pass through any discrete charge. This is because electric field due to a system of discrete charges is not well defined at the location of any charge. (As you go close to the charge, the field grows without any bound.) However, the Gaussian surface can pass through a continuous charge distribution.*
5- *Gauss's law is often useful towards a much easier calculation of the electrostatic field when the system <u>has some symmetry</u>. This is facilitated by the choice of a suitable Gaussian surface.*
6- *Finally, Gauss's law is based on the inverse square dependence on distance contained in the Coulomb's law. Any violation of Gauss's law will indicate departure*
7- *The surface should be chosen in such a way that at every point of surface electric field strength is either parallel or perpendicular to the surface.*
8- *Flux through Gaussian surface is independent of its shape.*
9- *Flux through Gaussian surface depends only on total charge present inside Gaussian surface.*
10- *Flux through Gaussian surface is independent of position of charges inside Gaussian surface.*
11- *Flux due to field lines entering a closed surface is taken as negative and flux due to field lines leaving a surface is taken as positive. This is because $\vec{\boldsymbol{n}}$ is taken positive in outward direction.*
12- *In a Gaussian surface $\Phi = \mathbf{0}$ does not imply $\boldsymbol{E} = \mathbf{0}$ at every point of the surface but $\boldsymbol{E} = \mathbf{0}$ at every point implies $\Phi = \mathbf{0}$.*

Surface S_1

The electric field is outward for all points on the surface. Thus, the flux of electric field through this surface is positive, and so is the net charge within the surface

Surface S_2

The electric field is inward for all points on the surface. Thus, the flux of electric field through this surface is negative, and so is the enclosed charge, as Gauss' law requires

Surface S_3

This surface encloses no charge ($Q_{enc} = 0$). Gauss' law requires that the net flux of the electric field through this surface be zero. This is reasonable, because the number of field lines entering from one side leave it from the other side.

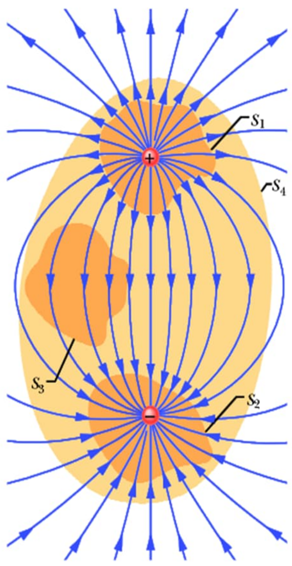


Fig.56

Surface S_4

This surface encloses no net charge ($Q_{enc} = 0$), because the enclosed positive and negative charges have equal magnitudes. Gauss' law requires that the net flux of the electric field through this surface be zero. This is reasonable, because there are as many field lines entering the surface as leaving it.

Worked examples

Example - 01

What is the field in all point of space due to a positive charged particle?

The particle is a point charge, so we have a spherical symmetry. Thus, Gauss law is powerful to calculate an electric field at any point of space.

Let a point A where we want to calculate the field at distance r from the charged particle.

The convenient Gaussian surface is a sphere (spherical symmetry) whose radius is the distance r from the charged particle, taken as center, to the desired point which is on that surface.

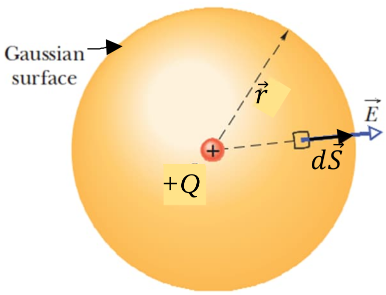


Fig.57

To calculate the flux through that surface we will define the direction of the element surface. In this case it is always outward in the radial direction. The same direction is that of the field at a desired point.

$$\vec{dS} = dS\,\vec{n} \qquad \vec{u}_r = \vec{n} \qquad \vec{E} = E\,\vec{u}_r$$

By applying the Gauss' law

$$\Phi = \iint_{SG} \vec{E} \circ d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

SG: is the Gaussian surface (sphere)

Q_{enc}: is the charge inside the Gaussian sphere

$\boldsymbol{E}$: The magnitude of the electric field is constant on the Gaussian surface. Because it depends only on the distance from the charge, which is the radius of the sphere.

$$\Phi = \iint_{SG} \vec{E} \circ d\vec{S} = \iint_{SG} E\, dS \cos(\vec{E}, d\vec{S})$$

$$\vec{E} \parallel d\vec{S} \qquad \Longrightarrow \quad \cos(\vec{E}, d\vec{S}) = 1$$

$$\iint_{SG} E\, dS \cos(\vec{E}, d\vec{S}) = \iint_{SG} E\, dS = E \iint_{SG} dS = E\, S$$

$\boldsymbol{S}$ is the surface of sphere: $S = 4\pi\, r^2$ and $Q_{enc} = +Q$

$$E\, S = E\,(4\pi\, r^2) = +Q/\epsilon_0 \quad \Longrightarrow \quad E = \frac{Q}{4\pi\, \epsilon_0 r^2}$$

Finally, because the electric field is in the radial direction its well defined

$$\vec{E} = \frac{Q}{4\pi\, \epsilon_0 r^2} \vec{u}_r$$

Example - 02

A long positively charged wire having a linear charge density $\boldsymbol{\lambda}$ uniformly distributed. By using Gauss's law, calculate the electric field strength at a point $\boldsymbol{P}$ located at distance $\boldsymbol{r}$ from the wire.

The convenient Gaussian surface is a cylinder (cylindrical symmetry) surrounding the line charge with Hight $\boldsymbol{l}$ and radius $\boldsymbol{r}$ which is the distance from the charged wire, taken as axis, to the desired point $\boldsymbol{P}$ which is on that surface.

The closed surface is composed by the lateral surface II, the upper surface I and the lower surface III

Since the distributed charge has a cylindrical symmetry, the field is in the plane of symmetry. i.e., it is oriented in the radial direction $\vec{E} = E\, \vec{u}_r$.

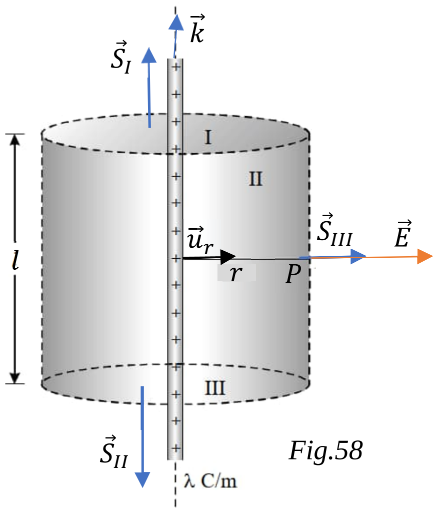


Fig.58

Gauss' Law:

$$\Phi = \iint_{SG} \vec{E} \circ d\vec{S} = \iint_{S_I} \vec{E} \circ d\vec{S} + \iint_{S_{II}} \vec{E} \circ d\vec{S} + \iint_{S_{III}} \vec{E} \circ d\vec{S}$$

$$d\vec{S}_I = dS\,\vec{k} \qquad d\vec{S}_{II} = dS\,\vec{u}_r \qquad d\vec{S}_{III} = -\,dS\,\vec{k}$$

$$\Phi = \iint_{S_I} E\,dS\cos(\frac{\pi}{2}) + \iint_{S_{III}} E\,dS\cos(-\frac{\pi}{2}) + \iint_{S_{II}} E\,dS\cos(0)$$

On the surface of the Gaussian surface (cylinder) the magnitude of the field is constant, so the flux through the closed surface is:

$$\Phi = \iint_{S_{II}} E\,dS = E\iint_{S_{II}} dS = ES_{II}$$

The enclosed charge for that surface is

$$Q_{enc} = \lambda\,l$$

The lateral surface S_{II} *has the area:*

$$S_{II} = 2\,\pi\,l$$

Finally, the Gauss' Law gives:

$$ES_{II} = E(2\,\pi\,l) = \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda\,l}{\epsilon_0}$$

$$\vec{E} = \frac{\lambda}{2\,\pi\epsilon_0}\vec{u}_r$$

ABOUT SYMMETRY

We often encounter systems with various symmetries. Consideration of these symmetries helps one arrive at results much faster than otherwise by a straightforward calculation. Consider, for example an infinite uniform sheet of charge (surface charge density σ*) along the y-z plane. This system is unchanged if*

(a) - translated parallel to the y-z plane in any direction,

(b) - rotated about the x-axis through any angle.

As the system is unchanged under such symmetry operation, so must its properties be. In particular, in this example, the electric field E must be unchanged.

Translation symmetry along the y-axis shows that the electric field must be the same at a point $(0, y_1, 0)$ *as at* $(0, y_1, 0)$*. Similarly translational symmetry along the z-axis shows that the electric field at two point (0, 0,* z_1*) and (0, 0,* z_2*) must be the same. By using* *rotation symmetry around the x-axis, we can conclude that E must be perpendicular to the y-z plane, that is, it must be parallel to the x-direction.*

Try to think of a symmetry now which will tell you that the magnitude of the electric field is a constant, independent of the x-coordinate. It thus turns out that the magnitude of the electric field due to a uniformly charged infinite conducting sheet is the same at all points in space. The direction, however, is opposite of each other on either side of the sheet.

Compare this with the effort needed to arrive at this result by a direct calculation using Coulomb's law.

As mentioned earlier, Gauss's law is useful for determining electric fields when the charge distribution is highly symmetric. The following examples demonstrate ways of choosing the gaussian surface over which the surface integral can be simplified and the electric field determined. In choosing the surface, always take advantage of the symmetry of the charge distribution so that E can be removed from the integral. The goal in this type of calculation is to determine a surface for which each portion of the surface satisfies one or more of the following conditions:

GAUSS LAW IN DIFFERENTIAL FORM

In the above we have used the GAUSS' Law in its integral form, what is its differential form that we will deduce in the next.

By using the divergence theorem, which states that the flux of any vector through any closed surface is equal to the integral of its divergence on the volume delimited by this closed surface.

Let $\vec{E}$ be an electrostatic field due to certain charge Q

$$\oiint \vec{E} \circ d\vec{S} = \iiint \vec{\nabla} \circ \vec{E}\, dv$$

GAUSS' Law in form integral is:

$$\Phi = \oiint \vec{E} \circ d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

But the enclosed charge is given by

$$Q_{enc} = \iiint \rho\, dv$$

Then

$$\oiint \vec{E} \circ d\vec{S} = \frac{\iiint \rho\, dv}{\epsilon_0} = \iiint \frac{\rho}{\epsilon_0}\, dv = \iiint \vec{\nabla} \circ \vec{E}\, dv$$

Finally, we deduce that

$$\frac{\boldsymbol{\rho}}{\boldsymbol{\epsilon_0}} = \vec{\boldsymbol{\nabla}} \circ \vec{\boldsymbol{E}}$$

Which is the GAUSS' Law in its differential form

LAPLACE EQUATION AND POISSON EQUATION

The procedure for determining the electric field E in the preceding chapters has generally been to use either Coulomb's law or Gauss's law when the charge distribution is known, or when the potential V is known throughout the region. In most practical situations, however, neither the charge distribution nor the potential distribution is known. In, we shall consider practical electrostatic problems where only electrostatic conditions (charge and potential) at some boundaries are known and it is desired to find E and V throughout the region. Such problems are usually tackled using Poisson's or Laplace's equation or the method of images, and they are usually referred to as boundary value problems. The concepts of resistance and capacitance will be covered. We shall use Laplace's equation in deriving the resistance of an object and the capacitance of a capacitor.

7 CONDUCTORS AND CAPACITANCE

7 -1 CONDUCTORS AND THEIR PROPERTIES

Conductors (such as metals) possess a large number of free electrons. They are free within the metal but not free to leave the metal. If there is an electric field, even is very weak, in the conductor these free charges will experience a force which will set a current flow. The free electrons form a kind of 'gas'; they collide with each other and with the ions, and move randomly in different directions. In an external electric field, they drift against the direction of the field. When no current flows, the resultant force and the electric field must be zero. Thus, under this, the conductor is in electrostatic equilibrium the value of E at all points within a conductor is zero. In contrast insulators such as glass or paper are materials in which electrons are attached to some particular atoms and cannot move freely

- *In metallic conductors, electrons of outer shells of the atoms are the free charges while the immobile positive ions are the bound charges.*
- *In electrolytic conductors, both positive and negative ions are the free charges.*
- *In insulators, both electrons and the positive ions are the bound charges.*
-

7 -1- 1 CONDUCTORS PROPERTIES

When there is no net motion of charge within a conductor, the conductor is in electrostatic equilibrium. A conductor in electrostatic equilibrium has the following properties:

1 - *Inside a conductor, electrostatic field is zero*

Consider a conductor, neutral or charged. There may also be an external electrostatic field $\vec{E}_0$. In the static situation, when there is no current inside or on the surface of the conductor, the electric field $\vec{E}$ is zero everywhere inside the conductor. This fact can be taken as the defining property of a conductor. A conductor has free electrons. As long as electric field is not zero, the free charge carriers would experience force and drift. In the static situation (electrostatic equilibrium), the free charges have so distributed themselves that the net electric field is zero everywhere inside. The charges induced create a field $\vec{E}_{in}$ which opposes the external field. The drift of charges continues until the induced electric field cancel the external field and we reach the electrostatic equilibrium

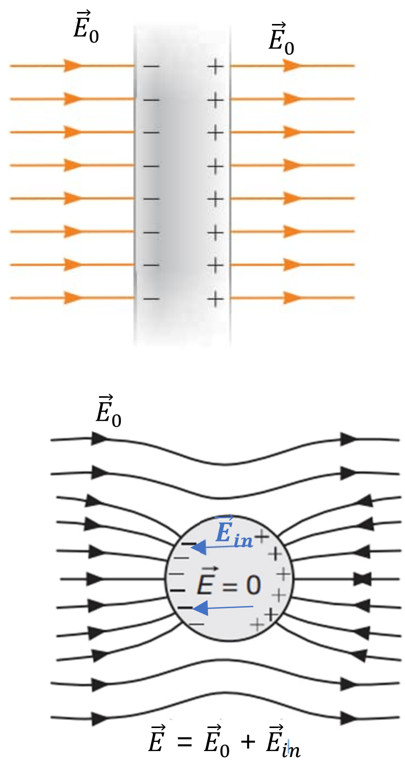


Fig.59

2 - *The interior of a conductor can have no excess charge in the electrostatic equilibrium*

A neutral conductor has equal amounts of positive and negative charges in every small volume or surface element. When the conductor is charged, the excess charge can reside only on the surface in the electrostatic equilibrium.

This follows from the Gauss's law. Consider any arbitrary volume element v inside a conductor. On the closed surface S bounding the volume element v, electrostatic field is zero. Thus, the total electric flux through S is zero. Hence, by Gauss's law, there is no net charge enclosed by S. But the surface S can be made as small as you like, i.e., the volume v can be made vanishingly small. This means there is no net charge at any point inside the conductor, and *any excess charge must reside at the surface*.

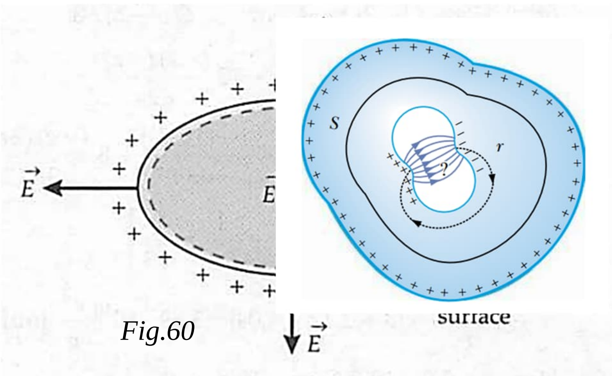


Fig.60

$$\Phi = \iint_{SG} E\, dS = 0 = \frac{Q_{enc}}{\varepsilon_0} \qquad \Rightarrow \qquad Q_{enc} = 0$$

$\boldsymbol{Q_{enc}} = \mathbf{0}$ *means that there is no excess charge in the volume delimited by the surface SG. Then, that excess charge, if it exists, it resides on the surface of the conductor*

The field inside the cavity of conductor is also zero

3 - *At the surface of a charged conductor, electrostatic field must be normal to the surface at every point*

If $\vec{E}$ were not normal to the surface, it would have some non-zero component along the surface. Free charges on the surface of the conductor would then experience force and move. To be in the electrostatic equilibrium situation, there is no motion of charges therefore, $\vec{E}$ should have no tangential component. Thus, electrostatic field of a charged conductor must be normal to the surface at every point. (For a conductor without any surface charge density, field is zero even at the surface.)

4 - *Electric field at the surface of a charged conductor is given by:* $\vec{E} = (\sigma/\varepsilon 0)\,\vec{n}$
With σ, the surface charge density and $\vec{n}$ the unit vector normal to the surface in the outward direction.

To derive the result, choose a short cylinder as the Gaussian surface about any point P on the surface, as shown in Fig. The cylinder is partly inside and partly outside the surface of the conductor. It has a small area of cross section ΔS and negligible height.

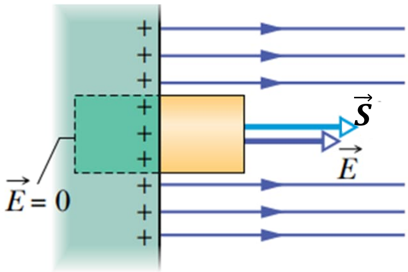


Fig.61

Just inside the surface, the electrostatic field is zero, just outside, the field is normal to the surface with magnitude E. Thus, the contribution to the total flux through the cylinder comes only from the outside (circular) cross-section of the cylinder. This, equals $\pm \boldsymbol{E}\,\Delta\boldsymbol{S}$ (positive for $\sigma > 0$, negative for $\sigma < 0$), since over the small area ΔS, $\boldsymbol{E}$ may be considered constant and are parallel or antiparallel. The charge enclosed by the cylinder is $\sigma.\Delta S$

By Gauss's law

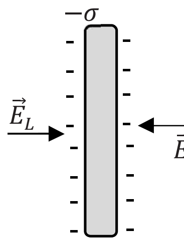
$$E\,.\Delta S = \frac{|\sigma|\,\Delta S}{\varepsilon_0}$$

$$E = \frac{|\sigma|}{\varepsilon_0}$$

This gives the magnitude of the field and from the property 3 that electric field is normal to the surface, we get the vector relation

$$\vec{E} = (\sigma/\varepsilon 0)\vec{n}$$

This relation is true for both signs of $\boldsymbol{\sigma}$. For $\boldsymbol{\sigma} > 0$, the electric field is normal to the surface outward and for $\boldsymbol{\sigma} < 0$, the electric field is normal to the surface inward.



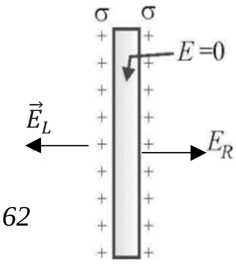


Fig.59

Fig.62

5 - *Electrostatic potential is constant throughout the volume of the conductor and has the same value (as inside) on its surface. The conductor is an equipotential volume*

This follows from the properties 1 and 2 above. Since $\vec{\boldsymbol{E}} = \vec{\boldsymbol{0}}$ inside the conductor and has no tangential component on the surface, no work is done in moving a small test charge within the conductor and on its surface. That is, there is no potential difference between any two points inside or on the surface of the conductor. Hence, the result. If the conductor is charged, electric field normal to the surface exists, this means potential will be different for the surface and a point just outside the surface.

6. *Electrostatic shielding*

Consider a conductor with a cavity, with no charges inside the cavity. A remarkable result is that the electric field inside the cavity is zero, whatever be the size and shape of the cavity and whatever be the charge on the conductor and the external fields in which it might be placed. We have proved a simple case of this result already: the electric field inside a charged spherical shell is zero. The proof of the result for the shell makes use of the spherical symmetry of the shell. But the vanishing of electric field in the (charge-free) cavity of a conductor is, as mentioned above, a very general result. A related result is that even if the conductor is charged or charges are induced on a neutral conductor by an external field, all charges reside only on the outer surface of a conductor with cavity.

Whatever be the charge and field configuration outside, any cavity in a conductor remains shielded from outside electric influence: the field inside the cavity is always zero. This is known as electrostatic shielding. The effect can be made use of in protecting sensitive instruments from outside electrical influence.

6. *The charge on the surface of the conductor are more concentrated in the shape of the high curvature (small radius of curvature)*

Let two conducting spheres, with radius $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ ($\boldsymbol{R_1} > \boldsymbol{R_2}$) charged on their surfaces with distributions $\boldsymbol{\sigma_1}$ and $\boldsymbol{\sigma_2}$ respectively, connected by a conductor wire (the system *constitutes a single conductor). From the property of equipotential, the surfaces of these two spheres and the wire are in the same potential* $V_2 = V_1$

We know that the potential of the sphere at any position r *is given by:*

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \quad \Longrightarrow \quad V_1 = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{r} \quad ; \quad V_2 = \frac{1}{4\pi\epsilon_0}\frac{Q_2}{r}$$

On the surface, these potentials become:

$V_1 = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R_1}$; $V_2 = \frac{1}{4\pi\epsilon_0}\frac{Q_2}{R_2}$

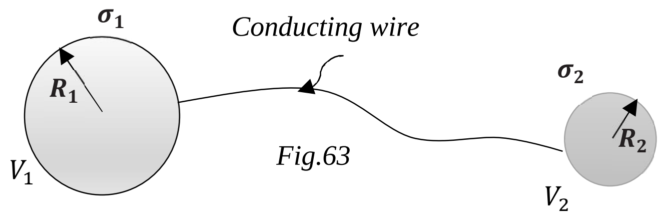


Fig.63

Since these spheres are conductor, the charge resides only on the surface

$Q_1 = \sigma_1 S_1 = 4\sigma_1\pi R_1{}^2$ *and* $Q_2 = \sigma_2 S_2 = 4\sigma_2\pi R_2{}^2$

So, $V_1 = \frac{1}{4\pi\epsilon_0}\frac{4\sigma_1\pi R_1{}^2}{R_1} = \frac{\sigma_1 R_1}{\epsilon_0}$*, and* $V_2 = \frac{1}{4\pi\epsilon_0}\frac{4\sigma_2\pi R_2{}^2}{R_2} = \frac{\sigma_2 R_2}{\epsilon_0}$

But all the system is in the same potential:

$$V_2 = V_1 \quad \Longrightarrow \quad \sigma_1 R_1 = \sigma_2 R_2$$

Now the ration $R_1/R_2 > 1$ $\quad \Longrightarrow \quad$ $\sigma_1 < \sigma_2$

The small sphere has more concentration of charges than the big sphere

The field in the vicinity of the small sphere is more intense than the big sphere. From the property of conductor $E = \frac{\sigma}{\varepsilon_0}$, $E_2 > E_1$

There a relation between the charge Q *of the conductor and the potential on its surface* V*. There is some proportionality, more the charge increased more the potential also.*

$$Q = C.V$$

C is the proportionality constant which we call capacitance

$$C = Q/V$$

For a spherical conducting sphere currying a charge Q*, the potential on its surface is:*

$$V = \frac{Q}{4\pi\varepsilon_0} \quad \Longrightarrow \quad C = Q/V = 4\pi\varepsilon_0$$

The earth is a big conductor with capacitance $C_e \approx 710\,\mu F$

Consider a parallel plate capacitor with plate area S*. Suppose a positive charge* $+Q$ *is given to one plate and negative charge* $-Q$ *to the other plate. The electric field due to the plate carrying positive charge is* $E = \sigma/2\varepsilon_0 = Q/(2S\varepsilon_0)$ *at all points if the plate is large.*

On the other hand, the plate carrying the negative charge finds itself in the field of the positive charge. Therefore, it experiences a force $\boldsymbol{F} = \boldsymbol{QE} = \boldsymbol{Q}^2/(2\boldsymbol{S}\boldsymbol{\varepsilon}_0)$, which is attractive.

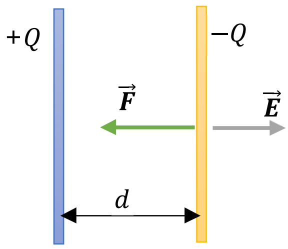


Fig.64

7 -1- 2 Energy Density (Energy per unit volume)

Consider a plate parallel capacitor, with S the area of the plate and d the plate separation. Let the charge on this capacitor be Q, then the electric field in the region between the plates is: $\boldsymbol{E} = \boldsymbol{\sigma}/\boldsymbol{\varepsilon}_0 = \boldsymbol{Q}/(\boldsymbol{S}\boldsymbol{\varepsilon}_0)$.

The volume of the capacitor is: $\boldsymbol{v} = \boldsymbol{S}.\boldsymbol{d}$

The energy stored in the field is:

Suppose that at a given instant, a charge q' has been transferred from one plate of capacitor to the other. The potential difference V' between the plates at that instant will be q'/C. If an extra increment of charge dq' is the transferred, the increment in the work required will be

$$dW = V'\,dq' = \frac{q'}{C}dq'$$

The work required to bring the total capacitor charge up to a final value Q is

$$W = \int_0^Q \frac{q'}{C}dq = \frac{1}{2}\frac{Q^2}{C}$$

This work is stored as potential energy $\boldsymbol{U}$ in the capacitor, so that

$$U = Q^2/2C = Q^2 d/2S\varepsilon_0 = (Q/S\varepsilon_0)^2\frac{1}{2}\varepsilon_0 S\,.\,d$$

The energy density is: $\boldsymbol{U}/\boldsymbol{v} = \boldsymbol{u} = \frac{1}{2}(\boldsymbol{Q}/\boldsymbol{S}\boldsymbol{\varepsilon}_0)^2\boldsymbol{\varepsilon}_0 = \frac{1}{2}\boldsymbol{\varepsilon}_0\boldsymbol{E}^2$

7 -1- 3 ELECTROSTATIC INFLUENCE

Let A be a neutral conductor, when it is placed in the electric field it will be polarized, creation a negative and a positive charges separately. The field lines end at the negative charge, then starts from the positive charge and go to infinity.

Let A be a conductor positively charged, it creates an electric field. It influences another neutral conductor B put near it.

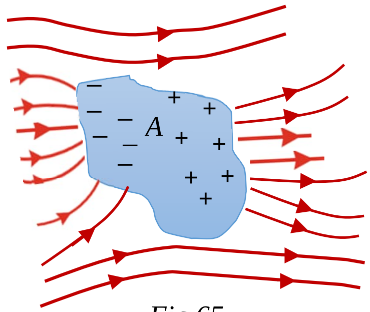


Fig.65

The conductor B, will be polarized, the facing side is negatively charged, while the opposite side is negatively charged. The surfaces of the facing side have the charges equal in magnitude but opposite.

Let take the tube with the side surfaces dS_1, dS_2 and the lateral surface dS_3. We apply the GAUSS' Law on this tube of lines of field which is a cylinder

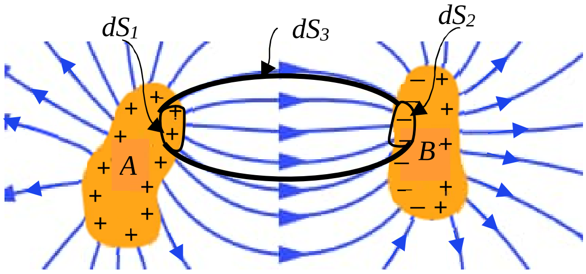


Fig.66

$$\Phi = \iint_{SG} \vec{E} \circ d\vec{S} = \iint_{S_I} \vec{E} \circ d\vec{S} + \iint_{S_{II}} \vec{E} \circ d\vec{S} + \iint_{S_{III}} \vec{E} \circ d\vec{S}$$

$$= \frac{Q_{enc}}{\varepsilon_0}$$

$$\vec{E} \circ d\vec{S}_1 = -E\, dS_1$$

$$\vec{E} \circ d\vec{S}_2 = E\, dS_2$$

$$\vec{E} \circ d\vec{S}_3 = 0$$

$$Q_{enc} = \sigma_A dS_1 + \sigma_B dS_2 = Q_A + Q_B$$

$$\Phi = \iint_{SG} \vec{E} \circ d\vec{S} = \iint_{S_I} -E\, dS_1 + \iint_{S_{II}} E\, dS_2 = 0 = \frac{Q_A + Q_B}{\varepsilon_0}$$

$$Q_A + Q_B = 0 \qquad \Rightarrow \qquad Q_B = -Q_A$$

Under the influence, the facing sides have equal and opposite charges

If all field lines starting from conductor of positive charge end to the conductor of negative charge, the influence is called total influence.

If some field lines starting from conductor of positive charge end to the conductor of negative charge, the influence is called partial influence

7 -2 CAPACITORS AND CAPACITANCE

Two conductors, called plates, under total influence and separated by an insulator, constitute a capacitor.

A capacitor is a device in which electric energy can be stored. A capacitor, in general, consists of two conductors of any size and shape carrying different potentials and charges, and placed closed together in some definite positions relative to one another. The advantage of capacitor compared to the battery, is that the capacitor can give us a high rate of energy compared to the battery which gives low rate.

Why we use two conductors to get a capacitor? Can we have a good capacitance for only an isolated conductor?

An isolated conductor cannot have a large capacitance. When a conductor holds a large amount of charge, its potential is also high, the associated electric field becomes high enough, the atoms or molecules of the surrounding air get ionized. A breakdown occurs in the insulation of the surrounding medium and the charge put on the conductor gets neutralized or leaks away. This limits the capacitance of a conductor. Moreover, if we tend to have a single conductor of large capacitance, it will have practically inconvenient large size.

To get a high capacitance, we use then, two conductors near each other. If one conductor is positively charged the other conductor is earthed (grounded), then the capacitance can be increased. (When an earthed conductor is placed near to a conductor then the capacitance of the conductor is greatly increased)

The charge on the capacitor is proportional to the potential difference between the two conductors

$$\boldsymbol{Q} = \boldsymbol{C}.\boldsymbol{V}$$

$$\boldsymbol{V} = \boldsymbol{V_2} - \boldsymbol{V_1} \quad \Rightarrow \quad \boldsymbol{C} = \frac{\boldsymbol{Q}}{\boldsymbol{V}}$$

How to calculate the capacitance?

1. *Assume a charge Q of the plates*
2. *Calculate the electric field* $\vec{\boldsymbol{E}}$ *in term of this charge Q between these two plates using the GAUSS' Law.*

$$\Phi = \iint_{SG} \vec{\boldsymbol{E}} \circ \boldsymbol{d}\vec{\boldsymbol{S}} = \frac{\boldsymbol{Q_{enc}}}{\boldsymbol{\varepsilon_0}}$$

3- Knowing the electric field $\vec{E}$, we calculate the potential difference between the two plates using the relations between the electric field and the potential.

$$V = V_2 - V_1 = -\int_1^2 \vec{E} \circ d\vec{l}$$

4- The capacitance is deduced from the quotient $\frac{Q}{V}$ between the charge and the potential difference

- Capacitance of parallel plates capacitor

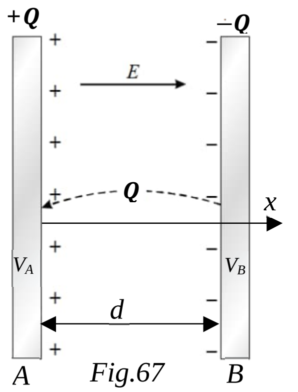


Fig.67

To calculate the capacitance, we follow the steps mentioned above

1- Let the charge of the plates $Q = \sigma A$

σ: Surface charge density

A: Area of the plates

2- For each plate the field is given by $E = \pm\sigma / 2\varepsilon_0$

The field in between the two plates is: $E = \sigma / \varepsilon_0 = Q/(\varepsilon_0 A)$

3- Calculate the potential difference between the plates

$$V = V_A - V_B = -\int_B^A \vec{E} \circ d\vec{l} = \frac{Q}{\varepsilon_0 A}(x_B - x_A) = \frac{Qd}{\varepsilon_0 A}$$

4- The capacitance can be deduced from the quotient $\frac{Q}{V}$

$$C = \frac{Q}{V} = \frac{Q}{V_A - V_B} = \varepsilon_0 \frac{A}{d}$$

The capacitance depends on the geometry of the plates and the electric properties (physics properties) of the separation media

- Spherical Capacitor

A Spherical Capacitor is a system of two concentric shells (or solid sphere surrounded by a concentric shell) as shown in figure-68. The radii of shells in this system are r_1 and r_2. To find the capacitance of this system we calculate the field between the shells then the potential difference and deduce this capacitance from the quotient Q/V.

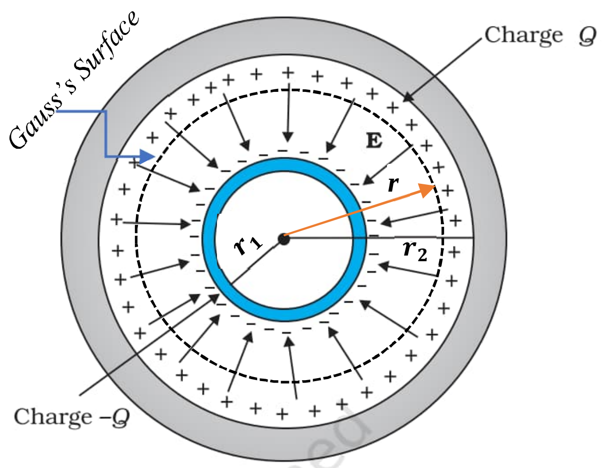


Fig.68

Let the charge of this capacitor be Q, to determine the field between these two conductors ($r_1 < r < r_2$), we use the GAUSS' Law

$$\Phi = \iint_{SG} \vec{E} \circ d\vec{S} = \frac{Q_{enc}}{\varepsilon_0}$$

For a spherical symmetry, the convenient Gauss's Surface is a sphere of radius $\boldsymbol{r}$

$$\iint_{SG} \vec{E} \circ d\vec{S} = E.S = E.(4\,\pi\, r^2) = -\frac{Q}{\varepsilon_0} \quad \Rightarrow \quad \vec{E} = -\frac{Q}{4\,\pi\,\varepsilon_0 r^2}\vec{u}_r$$

Now we calculate the potential difference between the conductors using the relation between the field and the potential

$$V = V_+ - V_- = -\int_{r_1}^{r_2} \vec{E} \circ d\vec{l} = \int_{r_1}^{r_2} \frac{Q}{4\,\pi\,\varepsilon_0 r^2}\vec{u}_r \circ d\vec{l} = \int_{r_1}^{r_2} \frac{Q}{4\,\pi\,\varepsilon_0 r^2} dr$$

$$V = \frac{Q}{4\,\pi\,\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{Q(r_2 - r_1)}{4\,\pi\,\varepsilon_0 r_1 r_2}$$

The capacitance can be deduced from the quotient between the charge Q and the potential difference V

$$C = \frac{Q}{V} = 4\,\pi\,\varepsilon_0 \frac{r_1 r_2}{r_2 - r_1}$$

- *Cylindrical Capacitor*

A system of two long coaxial cylindrical shells, is called Cylindrical Capacitor. As the shells are considered to be very long in this case, we analyze the capacitance per unit length of such a capacitor. To determine the capacitance of this system we transfer a charge Q from outer shell B to inner shell A due to which inner shell will gain a charge + Q and outer shell with a charge – Q which will be distributed on the inner surface of outer shell as shown in figure. In this case the electric field strength in the annular region between the two cylindrical shells is only due to the inner charge and it is in radially outward direction. The strength of electric field in this region at a distance r from the common axis is given using GAUSS' Law

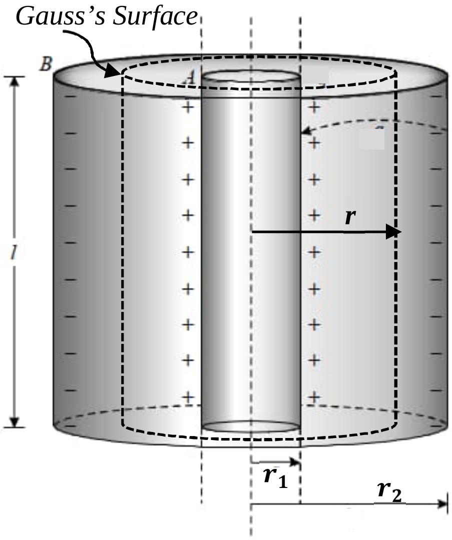


Fig.69

Cylindrical symmetry, the convenient Gauss's surface is a cylinder. The field is radial.

$$\iint_{SG} \vec{E} \circ d\vec{S} = E.S = E.(2\,\pi\, r\, l) = \frac{Q_{enc}}{\varepsilon_0} = \frac{\lambda\, l}{\varepsilon_0} \quad \Rightarrow \quad \vec{E} = \frac{\lambda}{2\,\pi\,\varepsilon_0\, r}\vec{u}_r$$

The potential difference between the conductors is found by using the relation between the field and the potential

$$V = V_+ - V_- = -\int_{r_2}^{r_1} \vec{E} \circ d\vec{l} = \int_{r_1}^{r_2} \frac{\lambda}{2\,\pi\,\varepsilon_0\, r} \vec{u}_r \circ d\vec{l}$$

$$V = \frac{\lambda}{2\,\pi\,\varepsilon_0} ln(r) \Big|_{r_1}^{r_2} = \frac{\lambda}{2\,\pi\,\varepsilon_0} ln\left(\frac{r_2}{r_1}\right)$$

The capacitance can be deduced from the quotient between the charge Q and the potential difference V

$$C = \frac{Q}{V} = \frac{2\,\pi\,\varepsilon_0\, l}{ln\left(\frac{r_2}{r_1}\right)}$$

The capacitance by unit of length is

$$C_l = \frac{2\,\pi\,\varepsilon_0}{ln\left(\frac{r_2}{r_1}\right)}$$

7 -2 COMBINATION OF CAPACITORS

We can combine several capacitors of capacitance C_1, C_2, ..., Cn to obtain a system with some effective capacitance C = C_{eq}. The effective capacitance depends on the way the individual capacitors are combined. Two simple possibilities are discussed below.

In a circuit, several such capacitors are often wired together and it is then necessary to calculate the net capacitance of the combination. The simplest ways of wiring capacitors together are in parallel and in series.

7 – 2 - 1 PARALLEL COMBINATION

In this combination the plates of the same sign of charge, not necessarily equal, are connected together. Then, the potential difference is same for all capacitors, but the whole charge in the system will distributed in dependance of capacity of each condenser.

Let illustrate with a system with two capacitors and then generalize to the situation when we have a several condensers connected in parallel

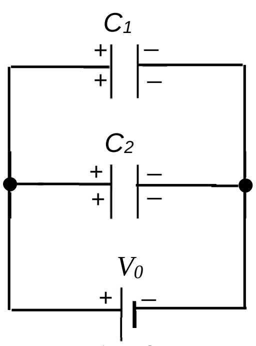


Fig.70 - a

When the two condensers are connected to voltage source V_0 (fig. 70-a), the charges in the left plates are distributed in such manner that their potentials are same as the positive potential of the source. Same thing happens in the right plates, their potentials *become equals to the negative potential of the battery. The potential differences for the two capacitors are equal to potential difference of the battery.*

$$V_o = V_+ - V_- = V_1 = V_2$$

The charge Q delivered by the source is shared by both capacitors, but not at equal proportion, it depends on the capacitance of each condenser

$$Q = Q_1 + Q_2$$

Q_1 *is the charge of the capacitor* C_1*:* $Q_1 = C_1V_1 = C_1V_0$

Q_2 *is the charge of the capacitor* C_2*:* $Q_2 = C_2V_2 = C_2V_0$

From the above equations

$$Q = C\,V_o = Q_1 + Q_2 = C_1V_0 + C_2V_0$$

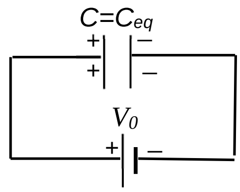


Fig.70 - b

The equivalent circuit is given by the fig. 70-b

Then the charge delivered by the source is $Q = C\,V_o = C_{eq}\,V_o$

We deduce that

$$C = C_{eq} = C_1 + C_2$$

If we have several capacitors combined in parallel and connected to the source. So, the potential differences of the condensers are same and the total charge is shared on all the capacitors, and we get the equivalent capacitance C_{eq} *as*

$$C = C_{eq} = C_1 + C_2 + \ldots + C_1 = \sum_{i=1}^{n} C_i$$

The net capacitance, or equivalent capacitance, of the parallel combination is simply the sum of the individual capacitances.

1- The equivalent capacitance is equal to the sum of the individual capacitances.

2- The equivalent capacitance is larger than the largest individual capacitance.

3- The potential difference across each capacitor is same

4- The charge on each capacitor is proportional to its capacitance.

7 – 2 - 1 SERIES COMBINATION

In this combination the passively charged is connected to the negative place of the second capacitor the remaining plates are connected to the battery

The charge in the right plate of C_1 induces a same but opposite charge on the left plate of the same capacitor. This charge creates the same opposite charge on the left plate of the second capacitor C_2 because the plates in the dashed line constitute a unique conductor

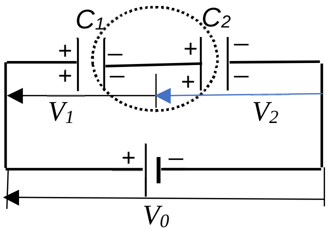


Fig.71 - a

In this configuration, the charge delivered by the source is same. Then

$\boldsymbol{Q = Q_1 = Q_2}$

The potential difference of the source is shared between the two capacitors but not in the same proportion. It depends on the capacitance of the condenser.

$$\boldsymbol{V_1 = Q_1/C_1 = Q/C_1}$$

$$\boldsymbol{V_2 = Q_2/C_2 = Q/C_1}$$

But

$$\boldsymbol{V_o = V_+ - V_- = V_1 + V_2}$$

The equivalent circuit is given by the fig. 71-b

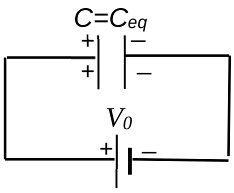


Fig.71 - b

Then the charge delivered by the source is $\boldsymbol{Q = C\,V_o = C_{eq}\,V_o}$

$$\boldsymbol{V_o = \frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_1}}$$

We deduce that

$$\boldsymbol{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_1}}$$

If we have a several condensers combined in series, then the equivalent capacitances given by the relation:

$$\boldsymbol{\frac{1}{C_{eq}} = \sum_{i=1}^{n} \frac{1}{C_i}}$$

1. *The reciprocal of equivalent capacitance is equal to the sum of the reciprocals of the individual capacitances.*

2. *The equivalent capacitance is smaller than the smallest individual capacitance.*

3. *The charge on each capacitor is same.*