MECHANICS OF A MATERIAL POINT (MATHEMATICAL REVIEW)

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1. Introduction

In the realm of physics, precision begins with understanding how to describe quantities that possess both magnitude and direction. This chapter is dedicated to exploring vectors and coordinate systems—the fundamental tools that allow us to represent physical quantities accurately in space. By mastering these concepts, we establish a solid foundation that will support our exploration of more complex topics in mechanics. Let's begin this journey by building the mathematical framework essential for delving deeper into the world of physics.

2. VECTORS

Physical Quantities

All quantities in terms of which laws of physics can be expressed and which can be measured are called Physical Quantities. There are scalar quantities and vector Quantities

Scalar quantity:

is always expressed by a numerical value followed by the corresponding unit.

Example: volume, mass, temperature

Vector Quantity:

A vector quantity is any quantity that requires a direction, a sense, a point of application, in addition to its numerical value called magnitude.

Example: displacement, velocity, force.

2.1. Notation and Graphical Representation

A vector is a mathematical entity that represents an element of a vector space associated with an affine space (of points) , where a direction, a sense, a magnitude, and a point of application are defined.

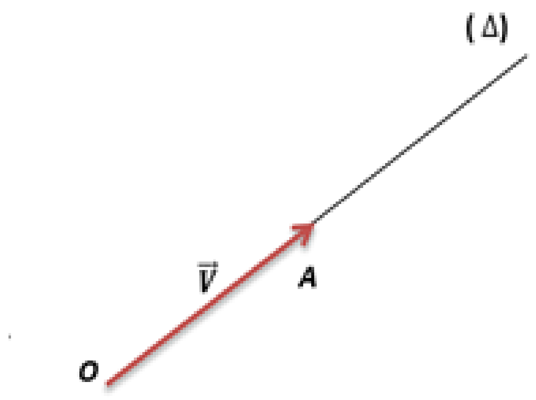
«**O**» Point of application.

«Δ» Direction (line of action).

«$||\overrightarrow{OA}||$»(in the orthonormal basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$.

$||\overrightarrow{OA}|| = \sqrt{x^2 + y^2 + z^2}$. And the magnitude of the vector.

From **O** to **A** is the direction.

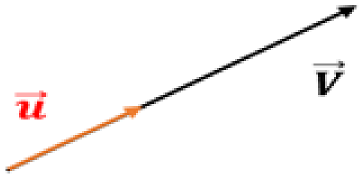


2.2. The unit vector

It is a vector with a magnitude equal to one (the number one).

A vector parallel to the unit vector can be expressed in the form:

$$\vec{V} = \vec{u}V = V\vec{u}$$



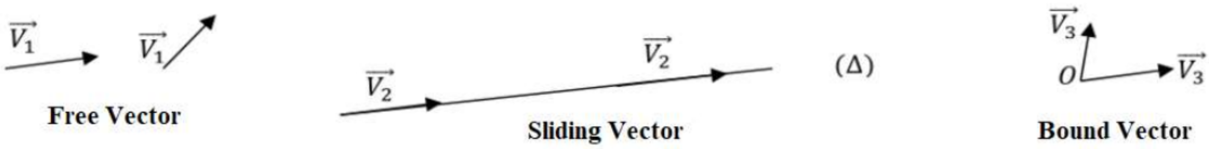
In general terms: Any vector that is parallel (in the same line) to another vector can be written as:

2.3. Types of Vectors

a) ***Free Vector:*** It is a vector where the point of application can be moved to any point in space.

b) ***Sliding Vector:*** It is a vector where the point of application can be moved along its line of action.

c) ***Bound Vector:*** It is a vector where the point of application is fixed and defined by the coordinates of its origin.

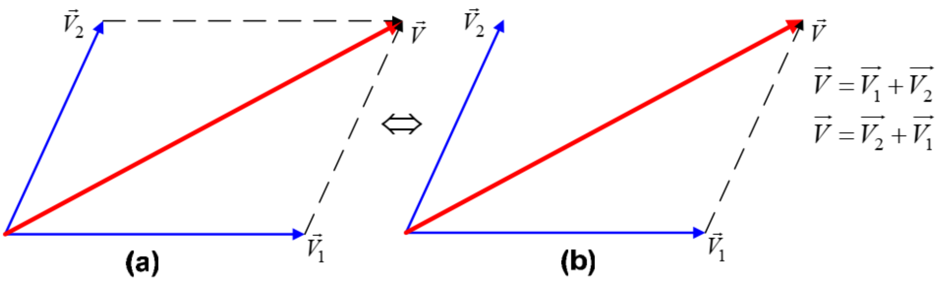


2.4. Vector Operations

The geometric sum of vectors

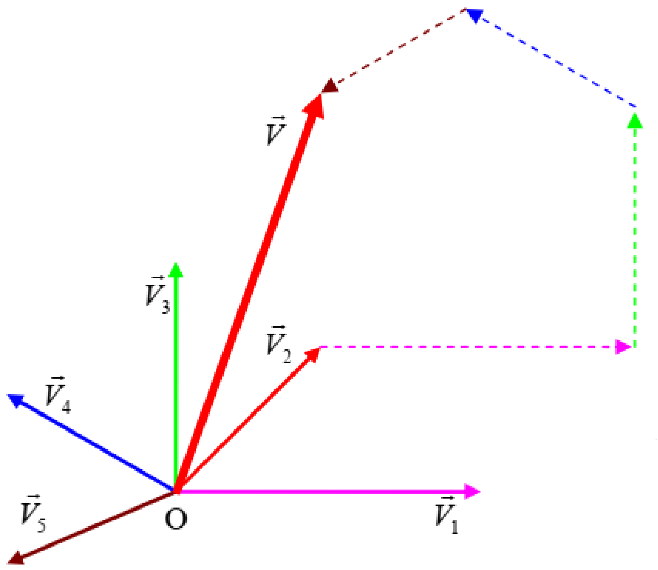
This operation relies on drawing, which is why it's called geometric.

The sum of two vectors is a commutative operation.



a) The geometric sum of several vectors

$$\vec{V} = \vec{V}_1 + \vec{V}_2 + \vec{V}_3 + \vec{V}_4 + \vec{V}_5$$



b) Law of Cosines

Relates the lengths of the sides of a triangle to the cosine of one of it angles, we can calculate the magnitude of the resulting vector using the law of cosines, which we will demonstrate later:

$$V = \sqrt{V_1^2 + V_2^2 + 2V_1V_2 \cos(\vec{V}1.\vec{V}1)}$$

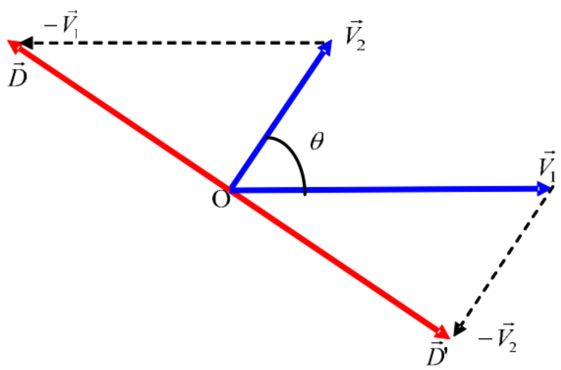
We can never write ~~$V = V_1 + V_2$ *(WRONG)*~~

c) Subtraction of Two Vectors

Geometrically, vector $\vec{D}$ represents the result of subtracting vector $\vec{V}_1$ from vector $\vec{V}_2$. We can write it as: $\vec{D} = \vec{V}_2 - \vec{V}_1$

This equation can also be written as: $\vec{D} = \vec{V}_2 + (-\vec{V}_1)$

Vector subtraction is anticommutative, as illustrated in Figure: $\vec{D}' = -\vec{D}$



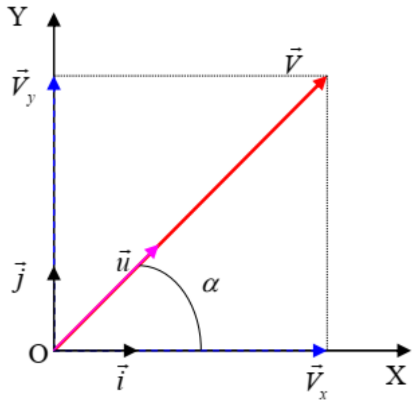
2.5. Components of a vector

Each vector can be considered as the sum of two or more vectors (the number of possibilities is unlimited).

In the plane, let the reference frame be denoted as $\mathcal{R}(O, \vec{i}, \vec{j})$

a) In rectangular coordinates

we decompose vector $\vec{v}$ along the X-axis and the Y-axis, as shown in Figure



By denoting the two unit vectors and , respectively in the directions of the two axes OX and OY, we can write:

$$\vec{V}_x = \vec{i}\, V_x,\ \vec{V}_y = \vec{j}\, V_y;$$

$$\vec{V} = \vec{V}_x + \vec{V}_y;\ \vec{V} = \vec{i}\, V_x + \vec{j}\, V_y$$

$$\vec{V} = \vec{i}\, V\cos\alpha + \vec{j}\, V\sin\alpha \rightarrow \vec{V} = V\,(\vec{i}\cos\alpha + \vec{j}\sin\alpha)$$

$$And\ \vec{V} = \vec{u}V$$

$$\vec{u} = \vec{i}\cos\alpha + \vec{j}\sin\alpha$$

As for the magnitude of vector $\vec{V}$, it is equal to: $V = \sqrt{V_x^2 + V_y^2}$

Using the coordinates *x* and *y*, we can also write: $V = \sqrt{x^2 + y^2}$

Example

Find the sum of the two vectors $\vec{V}_1\begin{pmatrix}x_1\\y_1\end{pmatrix}, \vec{V}_2\begin{pmatrix}x_2\\y_2\end{pmatrix}$ in the coordinate system $\mathscr{R}(O; \vec{i}, \vec{j})$.

Solution:

$$\begin{cases}\vec{V} = \vec{V}_1 + \vec{V}_2\\ \vec{V}_1 = x_1\vec{i} + y_1\vec{j} \rightarrow \vec{V} = (x_1\vec{i} + y_1\vec{j}) + (x_2\vec{i} + y_2\vec{j}) = = \vec{i}\,(x_1 + x_2) + \vec{j}\,(y_1 + y_2)\\ \vec{V}_2 = x_2\vec{i} + y_2\vec{j}\end{cases}$$

$$V = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

Example

Find the difference between the two vectors $\vec{V}_1\begin{pmatrix}x_1\\y_1\end{pmatrix}, \vec{V}_2\begin{pmatrix}x_2\\y_2\end{pmatrix}$ in the coordinate system $\mathscr{R}(O; \vec{i}, \vec{j})$.

Solution:

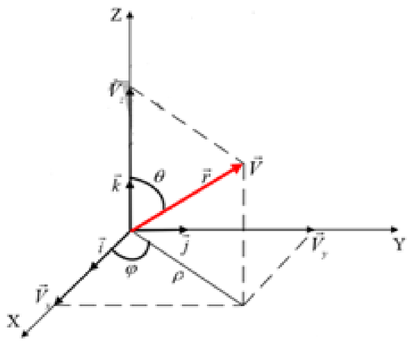
$$\begin{cases} \vec{V} = \vec{V}_1 - \vec{V}_2 \\ \vec{V}_1 = x_1\vec{i} + y_1\vec{j} \rightarrow \vec{V} = (x_1\vec{i} + y_1\vec{j}) - (x_2\vec{i} + y_2\vec{j}) = = \vec{i}\,(x_1 - x_2) + \vec{j}\,(y_1 - y_2) \\ \vec{V}_2 = x_2\vec{i} + y_2\vec{j} \end{cases}$$

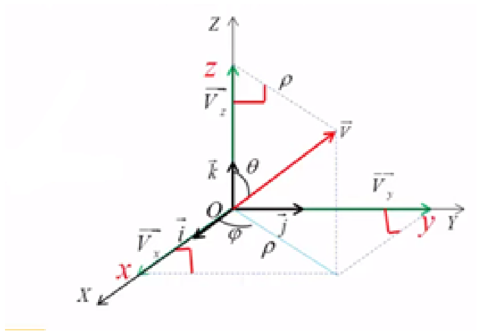
$$V = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

b) In space

in the coordinate system $\mathcal{R}(O; \vec{i}, \vec{j}, \vec{k})$ (orthonormal basis), we observe that:

$$\vec{V} = \vec{V}_x + \vec{V}_y + \vec{V}_z \rightarrow \vec{V} = \vec{i}\,V_x + \vec{j}\,V_y + \vec{k}\,V_z$$





We can geometrically ensure that:

$$\begin{cases} \cos\theta = \dfrac{V_Z}{r} \rightarrow V_Z = r.\cos\theta \\ \sin\theta = \dfrac{\rho}{r} \rightarrow \rho = r.\sin\theta \\ \cos\varphi = \dfrac{V_x}{\rho} \rightarrow V_x = \rho.\cos\varphi \rightarrow V_x = r.\sin\theta.\cos\varphi \\ \sin\varphi = \dfrac{V_y}{\rho} \rightarrow V_y = \rho.\sin\varphi \rightarrow V_y = r.\sin\theta.\sin\varphi \end{cases}$$

Then

$$\begin{cases} V_x = r.\sin\theta.\cos\varphi \\ V_y = r.\sin\theta.\sin\varphi \\ V_Z = r.\cos\theta \end{cases}$$

As for the magnitude of vector $\vec{V}$, it is equal to:

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

Or in Cartesian coordinates:

$$V = \sqrt{x^2 + y^2 + z^2}$$

Note: By denoting α and β as the respective angles formed by vector with the OX and OY axes, similar to how we obtained the previous equation, we have:

$$V_x = V.\cos\alpha, V_y = V.\cos\beta, V_z = V.\cos\theta$$

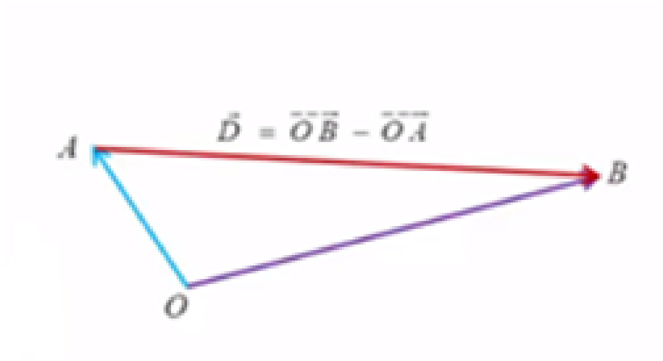
We can deduce the expression:

$$cos^2\alpha + cos^2\beta + cos^2\theta = 1$$

Example

Find the distance between the two points, going from **A** (10, -4, 4) *u* to **B** (10, 6, 8) *u*, represented in the rectangular coordinate system $\mathcal{R}(O; \vec{i}, \vec{j}, \vec{k})$, with ***u*** = the unit.

Solution: By representing the two points in the coordinate system, we realize that the distance in question is nothing other than the magnitude of vector $\vec{D}$, which is the difference between the two vectors: $\vec{D} = \vec{OB} - \vec{OA}$;



Hence:

$$\vec{D} = \vec{i}\,(x_2 - x_1) + \vec{j}\,(y_2 - y_1) + \vec{k}\,(z_2 - z_1) \rightarrow D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$D = \sqrt{116} = 10.77\ u$$

Example

Find the sum of the following five vectors:

$$\vec{V}_1 = (4\vec{i} - 3\vec{j})\ u;\ \vec{V}_2 = (-3\vec{i} + 2\vec{j})\ u;\ \vec{V}_3 = (2\vec{i} - 6\vec{j})\ u;\ \vec{V}_4 = (7\vec{i} - 8\vec{j})\ u;\ \vec{V}_5 = (9\vec{i} + \vec{j})\ u;$$

$$\vec{V} = (4 - 3 + 2 + 7 + 9)\ \vec{i} + (-3 + 2 - 6 - 8 + 1)\ \vec{j} = 19\ \vec{i} - 14\ \vec{j} \rightarrow V = \sqrt{361 + 196} = 23.60\ u$$

To find the direction of vector $\vec{V}$, we start with the expression $\tan\alpha = \frac{V_y}{V_x}$; α is the angle formed by vector $\vec{V}$ and the **OX** axis:

$$\tan\alpha = \frac{-14}{19} \approx -0.737 \rightarrow \alpha \approx -36.39^{o}$$

2.6. The dot product (The scalar product)

Az Definition

The scalar product of two vectors $\vec{V}_1$ and $\vec{V}_2$ is the real number:

$$\vec{V}_1 . \vec{V}_2 = V_1 . V_2 . \cos(\vec{V}_1 . \vec{V}_2)$$

$$\vec{V}_1 . \vec{V}_2 = \frac{1}{2}\left[\|\vec{V}_1 + \vec{V}_2\|^2 - \|\vec{V}_1\|^2 - \|\vec{V}_2\|^2\right]$$

Special case:

$$\text{If: } \vec{V}_1 = 0 \text{ or } \vec{V}_2 = 0, \text{ then } \vec{V}_1.\vec{V}_2 = 0.$$

$$\text{If: } \vec{V}_1 \neq 0 \text{ and } \vec{V}_2 \neq 0, \text{ then:}$$

$$\vec{V}_1 \perp \vec{V}_2 \rightarrow (\vec{V}_1.\vec{V}_2) = \frac{\pi}{2} \rightarrow \cos\frac{\pi}{2} = 0 \rightarrow \vec{V}_1.\vec{V}_2 = 0.$$

$$\vec{V}_1 // \vec{V}_2 \rightarrow (\vec{V}_1.\vec{V}_2) = 0 \rightarrow \cos 0 = 1 \rightarrow \vec{V}_1.\vec{V}_2 = V_1.V_2.$$

Example

The work done by the force $\vec{F}$ that causes a displacement $\overrightarrow{AB}$ is given by the formula $W = F{\cdot}AB{\cdot}\cos\alpha$, , where $\alpha = (\vec{F};\overrightarrow{AB})$ (we read W is the scalar (dot) product of $\vec{F}$ and $\overrightarrow{AB}$), , we write:

$$\vec{W} = \vec{F}.\overrightarrow{AB} \leftrightarrow W = \vec{F}.\overrightarrow{AB}.\cos\alpha$$

Let's now demonstrate

$$\vec{V} = \vec{V}_1 + \vec{V}_2;\ \vec{V}^2 = \vec{V}_1^2 + \vec{V}_2^2 + 2\vec{V}_1\vec{V}_2;\ \vec{V}_1^2 = \vec{V}_1\vec{V}_1 = V_1V_1\cos(\vec{V}_1.\vec{V}_1) = V_1^2$$

$$V^2 = V_1^2 + V_2^2 + 2V_1V_2\cos(\vec{V}_1.\vec{V}_1) \rightarrow V = \sqrt{V_1^2 + V_2^2 + 2V_1V_2\cos(\vec{V}1.\vec{V}1)}$$

a) Analytical Expression of the Dot Product

In the plane: Consider the two vectors $\vec{V}_1$ and $\vec{V}_2$ contained in the plane, such that:

$$\vec{V}_1\begin{pmatrix}x_1\\y_1\end{pmatrix}, \vec{V}_2\begin{pmatrix}x_2\\y_2\end{pmatrix}$$

In the coordinate system $\mathscr{R}(O;\vec{i},\vec{j})$.

$$\vec{V}_1\vec{V}_2 = (x_1.\vec{i} + y_1.\vec{j}).(x_2.\vec{i} + y_2.\vec{j}) = x_1.x_2.\vec{i}.\vec{i} + x_1.y_2.\vec{i}.\vec{j} + x_2.y_1.\vec{j}.\vec{i} + y_1.y_2.\vec{j}.\vec{j}$$

$$\vec{i} \perp \vec{j} \rightarrow \vec{j}.\vec{i} = \vec{i}.\vec{j} = 0$$

$$\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{i}^2 = \vec{j}^2 = 1 \qquad \vec{V}_1\vec{V}_2 = x_1.x_2 + y_1.y_2$$

In space: Consider the two vectors $\vec{V}_1$ and $\vec{V}_2$ in the coordinate system $\mathscr{R}(O;\vec{i},\vec{j},\vec{k})$

$$\vec{i}.\vec{j} = \vec{i}.\vec{k} = \vec{j}.\vec{k} = 0;\ \vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$$

$$\vec{V}_1\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}; \vec{V}_2\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix} \rightarrow \vec{V}_1\vec{V}_2 = x_1.x_2 + y_1.y_2 + z_1.z_2$$

b) Properties of the Dot Product

Commutative $\vec{V}_1\vec{V}_2 = \vec{V}_2\vec{V}_1$

Non-associative $\vec{V}_1.(\vec{V}_2.\vec{V}_3)$ Does not exist because the result would be a vector.

Distributive with respect to vector addition: $\vec{V}_1.(\vec{V}_2+\vec{V}_3) = \vec{V}_1\vec{V}_2 + \vec{V}_1\vec{V}_3$

Example

Calculate the angle between the two vectors: $\vec{V}_1 = 3\vec{i}+2\vec{j}-\vec{k}$ and $\vec{V}_2 = -\vec{i}+2\vec{j}+3\vec{k}$

Solution: Starting from the expression of the dot product, we can write:

$$\cos(\vec{V}_1.\vec{V}_2)=\frac{\vec{V}_1\vec{V}_2}{V_1V_2}$$

$$Then: \vec{V}_1\vec{V}_2=-3+4-3=-2;\ V_1=\sqrt{9+4+1}=3.74;\ V_2=\sqrt{1+4+9}=3.74$$

$$\cos(\vec{V}_1.\vec{V}_2)=\frac{\vec{V}_1\vec{V}_2}{V_1V_2}=\frac{-2}{14}=-0.143 \rightarrow \theta=(\vec{V}_1.\vec{V}_2)=96.2°$$

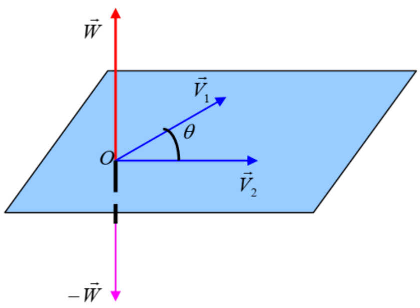
2.7. The cross product (vector product)

Az Definition

The cross product of two vectors $\vec{V}_1$ and $\vec{V}_2$ is the vector $\vec{W}$ that is perpendicular to the plane formed by $\vec{V}_1$ and $\vec{V}_2$.

We write it conventionally as:

$$\vec{W} = \vec{V}_1 \wedge \vec{V}_2 = \vec{V}_1 \times \vec{V}_2$$



a) Characteristics of vector $\vec{W}$

- $\vec{W}$ is perpendicular to the plane formed by the two vectors.
- Its direction is determined by the right-hand rule, with the index finger pointing in the direction of **W**.
- Its magnitude is given by formula

$$\vec{V}_1 \wedge \vec{V}_2 = \|\vec{V}_1\|.\|\vec{V}_2\|\sin(\vec{V}_1;\ \vec{V}_2)\vec{u}$$

Important:

$$\vec{\imath} \wedge \vec{\jmath} = \vec{k} \quad \vec{\jmath} \wedge \vec{k} = \vec{\imath} \quad \vec{k} \wedge \vec{\imath} = \vec{\jmath} \quad \vec{\imath} \wedge \vec{\imath} = \vec{\jmath} \wedge \vec{\jmath} = \vec{k} \wedge \vec{k} = \vec{0}$$

> 💬 Note
>
> The quantity $\vec{V_1} \wedge \vec{V_2} = \|\vec{V_1}\|.\|\vec{V_2}\| \sin(\vec{V_1}; \vec{V_2})\vec{u}$ represents the area of the parallelogram formed by the two vectors. This suggests the possibility of associating a vector with a certain surface.

b) Method used to calculate the cross product of two vectors

$$\vec{V_1}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} ; \vec{V_2}\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

By using Cartesian coordinates in the coordinate system $\mathscr{R}(O; \vec{\imath}, \vec{\jmath}, \vec{k})$, we can write:

$$\vec{V_1} \wedge \vec{V_2} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \vec{\imath} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \vec{k}$$

$$\vec{W} = (y_1 z_2 - y_2 z_1)\, \vec{i} - (x_1 z_2 - x_2 z_1)\vec{j} + (x_1 y_2 - x_2 y_1)\, \vec{k}$$

The magnitude of the vector is given by the expression:

$$\boxed{W = \sqrt{(y_1 z_2 - y_2 z_1)^2 + (x_1 z_2 - x_2 z_1)^2 + (x_1 y_2 - x_2 y_1)^2} = V_1.V_2.\sin(\vec{V_1}, \vec{V_2})}$$

c) Properties of the Cross Product

- The cross product is non-commutative: $\vec{V_1} \wedge \vec{V_2} = -\vec{V_2} \wedge \vec{V_1}$
- The cross product is distributive with respect to addition.

$$\vec{V_1} \wedge (\vec{V_2} + \vec{V_3}) = \vec{V_1} \wedge \vec{V_2} + \vec{V_1} \wedge \vec{V_3}$$

- The resulting vector from the cross product is always perpendicular to the operand vectors.
- The cross product follows the cyclic permutation.

$$\vec{\imath} \wedge \vec{\jmath} = \vec{k} \quad \vec{\jmath} \wedge \vec{k} = \vec{\imath} \quad \vec{k} \wedge \vec{\imath} = \vec{\jmath} \quad \vec{\imath} \wedge \vec{\imath} = \vec{\jmath} \wedge \vec{\jmath} = \vec{k} \wedge \vec{k} = \vec{0}$$

The cross product is zero if: $\|\vec{V_1}\| = 0$, $\|\vec{V_2}\| = 0$ or $\vec{V_1} \mathbin{/\!/} \vec{V_2}$

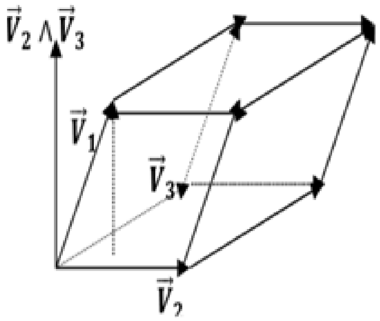
Note

The cross product geometrically represents the area of the oriented surface formed by the operand vectors.

2.8. The mixed product (Scalar Triple Product)

The mixed product of three vectors $\vec{V}_1, \vec{V}_2$, and $\vec{V}_3$ is the scalar quantity defined by:

$$\vec{V}_1.\left(\vec{V}_2 \wedge \vec{V}_3\right) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \left(y_2 z_3 - y_3 z_2\right) x_1 - \left(x_2 z_3 - x_3 z_2\right) y_1 + \left(x_2 y_3 - x_3 y_2\right) z_1$$



a) Properties of the Scalar Triple Product

- The scalar triple product is invariant under cyclic permutation.

$$\vec{V}_1.\left(\vec{V}_2 \wedge \vec{V}_3\right) = \vec{V}_3.\left(\vec{V}_1 \wedge \vec{V}_2\right) = \vec{V}_2.\left(\vec{V}_3 \wedge \vec{V}_1\right)$$

- The scalar triple product is zero if: $\|\vec{V}_1\| = 0$, $\|\vec{V}_2\| = 0$, $\|\vec{V}_3\| = 0$, , or $\vec{V}_1, \vec{V}_2$ and $\vec{V}_3$, are coplanar.
- The scalar triple product geometrically represents the volume formed by the operand vectors.

2.9. Gradient, Divergence, and Curl

Az Definition

- A function *f(x, y, z)* is called a scalar field if the function *f(x, y, z)* is a scalar.
- A function *(x, y, z)* is called a vector field if the function is vector-valued.
- We define the vector differential operator ∇ (nabla) as:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Where **∂/∂x, ∂/∂y**, and **∂/∂z** are the partial derivatives with respect to ***x, y***, and ***z***, respectively.

We will define the gradient, divergence, and curl using this operator.

a) The gradient

If *f(x, y, z)* is a scalar function, its gradient is a vector defined as:

$$\overrightarrow{grad}\,f = \vec{\nabla}(f) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

b) The divergence

If $\vec{V} = (Vx, Vy, Vz)$ is a vector function, its divergence is a scalar defined as:

$$div\vec{V} = \vec{\nabla}.\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

c) The curl

If $\vec{V} = (Vx, Vy, Vz)$ is a vector function, its curl is a vector defined as:

$$\overrightarrow{Curl(\vec{V})} = \vec{\nabla} \wedge \vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right).\vec{i} - \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right).\vec{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right).\vec{k}$$

i) Procedure to follow

a/ Establish the following matrix:

$$\overrightarrow{Curl(\vec{V})} = \begin{vmatrix} +\vec{i} & -\vec{j} & +\vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = A + B + C$$

b/ To calculate A, B, and C, simply remember the rule of the cross product:

$$A = \begin{vmatrix} +\vec{i} & & \\ & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ & V_y & V_z \end{vmatrix} = +\vec{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)$$

$$B = \begin{vmatrix} & -\vec{j} & \\ \frac{\partial}{\partial x} & & \frac{\partial}{\partial z} \\ V_x & & V_z \end{vmatrix} = -\vec{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)$$

$$C = \begin{vmatrix} & & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \\ V_x & V_y & \end{vmatrix} = +\vec{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

c/ We arrive at the final expression

$$\begin{vmatrix} +\vec{i} & -\vec{j} & +\vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = +\vec{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) -\vec{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \vec{k}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)$$

3. COORDINATE SYSTEM

In order to determine the instantaneous position of a material point, we must first choose a reference frame from among the various most useful frames. In the following, we will review the main coordinate systems.

3.1. Inertial or Galilean reference frames

Az Definition

(Galileo 1564-1642)

To determine the position of an object in space, we must first choose a solid body, which we call a reference frame, to which we associate coordinate axes.

Any set of coordinate axis systems, linked to a solid body S, which is the reference frame, constitutes a reference frame linked to this solid body S.

Example

The Earth (reference frame) + 3 axes, regardless of their common origin = a reference frame linked to the Earth.

In a given Galilean reference frame R, we locate a point position M using three spatial coordinates and one temporal coordinate, so the position is defined by four real numbers, for example **(X, Y, Z, t).**

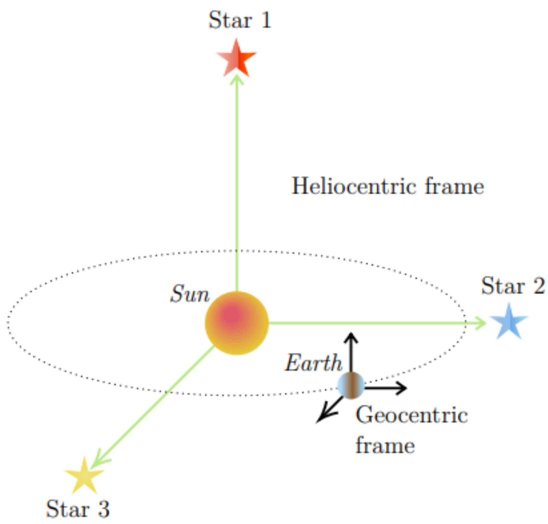
If we denote the position of a point M as **r = OM (x, y, z, t)** at time t, its motion in reference frame **R** is defined by the function **r(t)**.

3.2. Principal Galilean reference frames

a) Copernican Reference Frame

(Copernicus 1473-1543)

This reference frame is defined by three axes originating from the center of the solar system and directed toward three suitably chosen fixed stars. This system is used for the study of the motion of planets and interplanetary spacecraft. The Earth completes one rotation around the north-south pole in one day, and its revolution around the Sun takes one year.



b) Geocentric Reference Frame

This reference frame is defined by three axes originating from the center of gravity of the Earth and directed toward three fixed stars in the Copernican reference frame. This reference frame is used for the study of the motion of the Moon and satellites in rotation around the Earth.

c) Earth-Centered Reference Frame

This reference frame is defined by three perpendicular axes originating from any point on Earth. This reference frame is used for the study of bodies in motion related to the Earth. In this frame, the Earth is stationary, making it a Galilean reference frame.

3.3. Cartesian Coordinates

a) Spatial Frame

If the motion takes place in space, it is possible to locate the position of the point mass **M** in the reference frame $\mathscr{R}(O; \vec{i}, \vec{j}, \vec{k})$ using the position vector $\overrightarrow{OM}$ or using Cartesian coordinates (by René Descartes 1596-1650), which are:

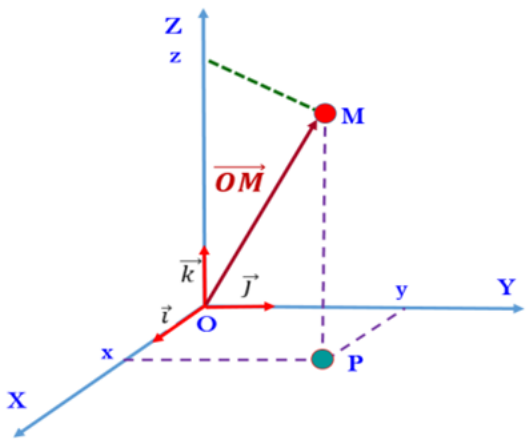
x: abscissa

y: ordinate

z: altitude

The position vector is then written as:

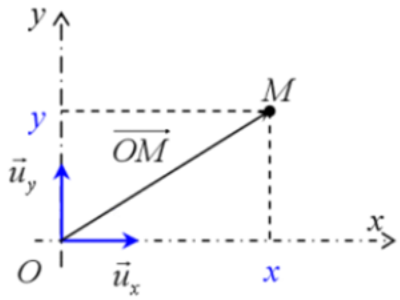
$$\boxed{\overrightarrow{OM} = \vec{r} = x.\vec{i} + y.\vec{j} + z.\vec{k}}$$



b) The Plane Frame

If the motion takes place in a plane, it is possible to locate the position of the point mass **M** in the reference frame $\mathscr{R}(O, \vec{i}, \vec{j})$ using rectangular coordinates **x** and **y**, or using the position vector $\overrightarrow{OM}$. The position vector is therefore written as:

$$\boxed{\overrightarrow{OM} = \vec{r} = x.\vec{i} + y.\vec{j}}$$



c) Rectilinear Frame

If the motion is rectilinear, we only need the **OX** axis, and the position vector $\overrightarrow{OM}$ is written as:

$$\boxed{\overrightarrow{OM} = \vec{r} = x.\vec{i}}$$

3.4. Polar Coordinates

If we choose a local basis $(\vec{u}_\rho, \vec{u}_\theta)$; **O** : arbitrarily selected as the pole, and $\vec{u}_\rho$ is oriented along the vector $\overrightarrow{OM}$.

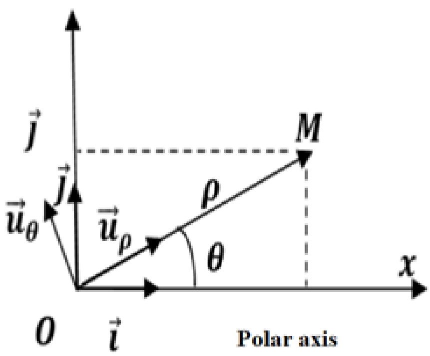
The direction that passes through the pole O is the polar axis; it is taken as a reference to define the angle (the coordinate) θ .

$$0 \leq \rho \leq +\infty \ , \ 0 \leq \theta \leq 2\pi$$

The other coordinate ρ is the magnitude of the vector $\overrightarrow{OM}$

$\overrightarrow{OM} = \rho\, \vec{u}_\rho$, The magnitude is: $\|\overrightarrow{OM}\| = \rho$

$$\begin{cases} \vec{u_\rho} = \cos\theta\, \vec{i} + \sin\theta\, \vec{j} \\ \vec{u_\theta} = -\sin\theta\, \vec{i} + \cos\theta\, \vec{j} \end{cases}$$

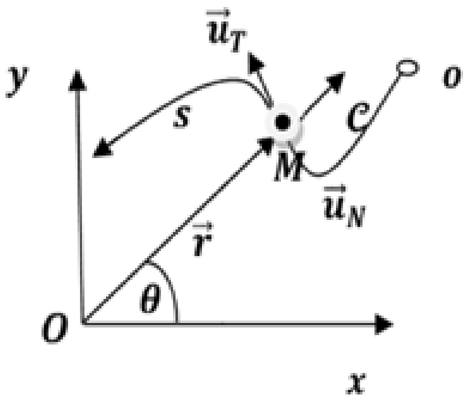


3.5. Intrinsic Coordinates

A point can only be represented in the system of intrinsic coordinates if we know the curve $\mathcal{C}$ of the trajectory taken as the axis.

Equipped with an origin, the distance $\widehat{OM}$ is denoted ***S***

$\widehat{OM} = s$ and $\overrightarrow{OM} = \vec{r}$



3.6. Relation Between Cartesian and Polar Coordinates

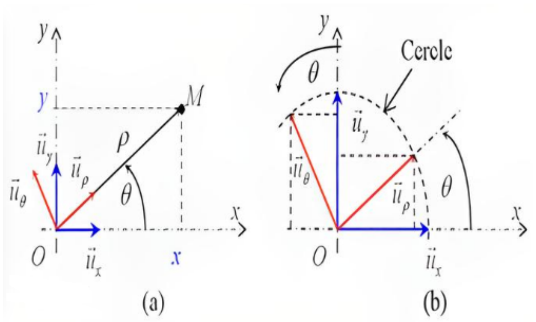
In Cartesian coordinates $\overrightarrow{OM} = \vec{r} = x\vec{\imath} + y\vec{\jmath}$

In polar coordinates $\overrightarrow{OM} = \rho\, \vec{u}_\rho$

If we make a choice such that the polar axis coincides with the axis $\overrightarrow{ox}$

$$\overrightarrow{OM} = \vec{r} = x\vec{\imath} + y\vec{\jmath} = \rho\vec{u_\rho} = \rho\cos\theta\vec{\imath} + \rho\sin\theta\vec{\jmath}$$

Furthermore, $\begin{cases} \vec{u_\rho} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} \\ \vec{u_\theta} = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath} \end{cases}$



By comparison, we will have

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Leftrightarrow \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = arct\ \frac{y}{x} \end{cases}$$

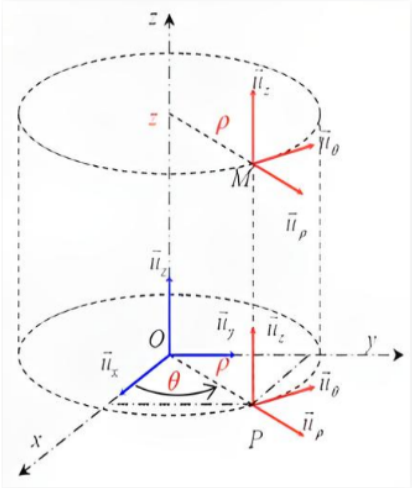
3.7. Cylindrical Coordinates

The point ***M*** is located on the surface of a cylinder. The projection of $\overrightarrow{OM}$ onto its base is identified by (ρ, θ, z).

$$0 \le \rho \le +\infty \quad 0 \le \theta \le 2\pi \quad 0 \le z \le +\infty$$

$$\overrightarrow{OM} = \vec{r} = \rho\vec{u_\rho} + z\vec{k}$$

And the magnitude is: $\|\overrightarrow{OM}\| = \|\vec{r}\| = \sqrt{\rho^2 + z^2}$



3.8. Relationship Between Cartesian and Cylindrical Coordinates

In Cartesian coordinates $\overrightarrow{OM} = \vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

In cylindrical coordinates $\overrightarrow{OM} = \vec{r} = \rho\overrightarrow{u_\rho} + z\vec{k}$

Furthermore, $\overrightarrow{u_\rho} = \cos\theta\vec{\imath} + \sin\theta\vec{\jmath}$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \Leftrightarrow \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = arct\,\frac{y}{x} \\ z = z \end{cases}$$

3.9. Spherical Coordinates

- The point ***M*** is located on the surface of a sphere.
- θ the polar angle: is the angle between the arbitrarily chosen polar axis and the direction $\overrightarrow{OM}$ where ***O*** is the center of this sphere.
- The projection $\overrightarrow{OM}$ onto the equatorial plane, is identified by the azimuthal angle $\boldsymbol{\varphi}$ relative to an arbitrarily chosen direction axis in this plane.

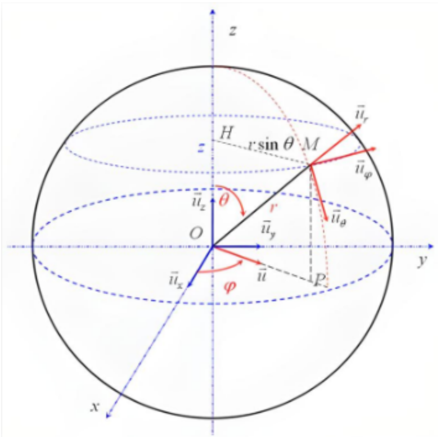
$$\overrightarrow{OM} = \vec{r} = \|\vec{r}\|\ \overrightarrow{u_r}$$

$$0 \le r \le +\infty \qquad 0 \le \theta \le \pi \qquad 0 \le \varphi \le 2\pi$$

$\vec{u}_r$: Unit vector in the radial direction (towards the radius $\overrightarrow{OM}$).

$\vec{u}_\theta$: Unit vector tangent to the great circle (all circles with radius $\overrightarrow{OM}$).

$\vec{u}_\varphi$: Unit vector tangent to the parallels (circles parallel to the equator).



3.10. Relationship Between Cartesian and Spherical Coordinates

In Cartesian coordinates $\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

In spherical coordinates $\overrightarrow{OM} = \vec{r} = \|\vec{r}\| \ \overrightarrow{u_r}$

Furthermore, $\overrightarrow{u_r} = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}$

Therefore

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\cos\theta \end{cases} \Leftrightarrow \begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \theta = arct\,\frac{y}{x} \\ z = arccos\,\frac{z}{r} \end{cases}$$

Conclusion

In Chapter 0, we revisited essential mathematical tools, particularly vectors and coordinate systems, which are foundational for understanding the mechanics of a material point. Vectors serve as a fundamental language for describing physical quantities, allowing us to represent and manipulate them effectively. By exploring different coordinate systems, including Cartesian, polar, and cylindrical, we've laid the groundwork for analyzing motion in various contexts. This mathematical review equips us with the necessary tools to tackle the complexities of physical phenomena, ensuring that we're well-prepared for the more intricate topics ahead.