MECHANICS OF A MATERIAL POINT (KINEMATICS OF A MATERIAL POINT)

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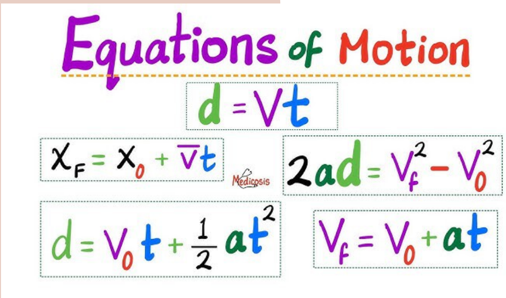

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1. Introduction

Kinematics is the study of motion without concern for the underlying causes responsible for that motion, such as forces, for example.

A material point is any material object with theoretically negligible dimensions compared to the distance it travels.

The state of motion or rest of an object is essentially relative. For example, a mountain is at rest relative to the Earth, but it is in motion relative to an observer who is looking at the Earth from a distance. From the perspective of this observer, the Earth (with everything it contains) is in constant motion.

Anyone wishing to study motion must, in advance, establish a reference frame (or a reference point) with respect to which the motion is analyzed. This means that motion can only be defined with respect to a reference frame.

The study of motion is conducted in one of two forms:

Vectorial: Using vectors such as position $\overrightarrow{OM}$, velocity $\vec{v}$, and acceleration $\vec{a}$.

Algebraic: Defining the equation of motion along a given trajectory.

2. *CHARACTERISTICS OF MOTION*

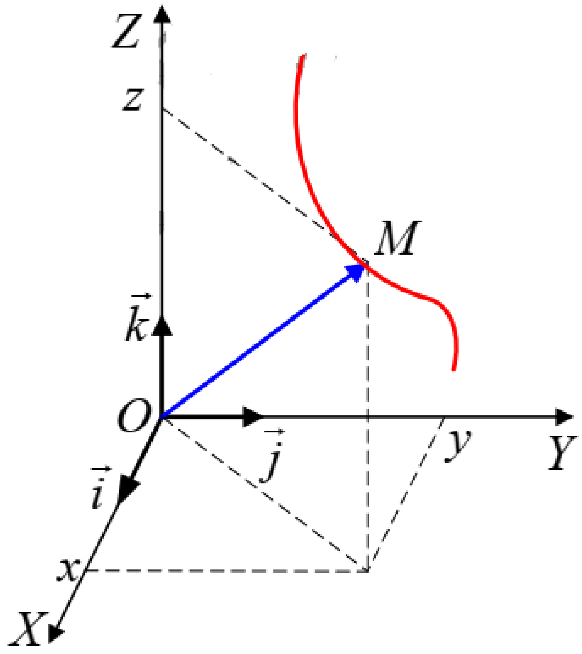
At the end of this chapter, the student will have acquired scientific skills related to:

- Describing and writing the motion of a material point in different coordinate systems,
- Understanding and being able to calculate the velocities and accelerations of a mobile for any trajectory,

2.1. *POSITION OF THE OBJECT*

The position of a material point ***M*** at time **(t)** is located in a reference frame $\mathcal{R}(O, \vec{i}, \vec{j}, \vec{k})$ by a position vector $\overrightarrow{OM}$ (Figure). Next formula expresses the position vector in Cartesian coordinates.

$$\overrightarrow{OM} = \vec{r} = x.\vec{i} + y.\vec{j} + z.\vec{k}$$



2.2. *TIME EQUATIONS*

A material point is at rest in a chosen reference frame if its coordinates **x, y, z** are independent of time, and it is in motion if its coordinates vary with time.

These coordinates can be noted as:

$$\boxed{x(t), y(t), z(t)}$$

These functions are called the time equations of motion. They can be expressed in the form:

$$\boxed{x = f(t), y = g(t), z = h(t)}$$

a) The Trajectory

The trajectory is the set of positions occupied by the object during its motion over successive moments. The trajectory can be physical (such as the path followed by a car) or imaginary (for example, the trajectory of the moon).

The study of planar motion is carried out in rectangular coordinates in the reference frame $\mathcal{R}(O, \vec{i}, \vec{j})$ where the position is defined by two coordinates: **x(t), y(t).**

The function **x** → **y(x)** is called the Cartesian equation of the trajectory.

The equation of the trajectory is obtained by eliminating time between the two time equations.

Example

The time equations for the motion of a material point projected into space are given as follows: **x= 2 t, y = 0, and z = -5 t^2+ 4 t**

1/ Find the Cartesian equation of the trajectory, and what is its form?

2/ Write the expression of the position vector at time **t = 2 s**.

Answer:

1/ We isolate ***t*** from the equation for ***x*** and then substitute it into the equation for ***z***:

$$x = 2t \Rightarrow t = \frac{x}{2} \quad \boxed{z = -1.25.x^2 + 2.x}$$

This is the equation of a parabola.

2/ Expression of the position vector:

At time **t = 2 s**, we have:

$$\overrightarrow{OM} = x.\vec{i} + y\vec{j} + z.\vec{k}$$

$$\overrightarrow{OM} = (2t).\vec{i} + (-5t^2 + 4t).\vec{k} \Rightarrow \overrightarrow{OM}_{(t=2)} = 4\vec{i} - 12\vec{k}$$

$$\boxed{\overrightarrow{OM}_{(t=2)} = 4\vec{i} - 12\vec{k}}$$

Example

The motion of a material point is defined in a Cartesian coordinate system by the following two time equations:

$$x = a\sin(\omega t + \varphi)$$
$$y = a\cos(\omega t + \varphi)$$

What is the trajectory followed?

Answer:

To find the trajectory, we square both equations and then add them together term by term. This leads to the equation of a circle with a radius "***a***"

$$\left.\begin{array}{l} x^2 = a^2\sin^2(\omega.t + \varphi) \\ y^2 = a^2\cos^2(\omega.t + \varphi) \end{array}\right| \Rightarrow x^2 + y^2 = a^2$$

2.3. *THE VELOCITY VECTOR*

Velocity is considered as the distance traveled per unit of time.

a) Average Velocity Vector

Let's look at the next Figure: between the instant ***t*** when the object is at position ***M*** and the instant ***t'*** when the object is at position ***M'***, the average velocity vector is defined as the expression.

$$\vec{v}_{AVR} = \frac{\overrightarrow{MM'}}{t' - t} \quad v_{AVR} = \frac{\left|\overrightarrow{MM'}\right|}{\Delta t}$$

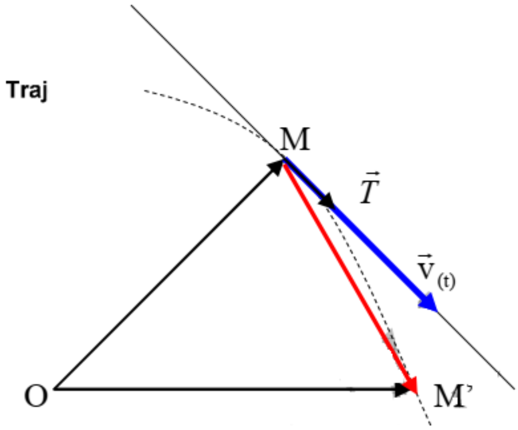
$\overrightarrow{MM'}$ called the displacement vector.

b) Instantaneous Velocity Vector

The instantaneous velocity vector, at time ***t***, is the derivative of the position vector with respect to time.

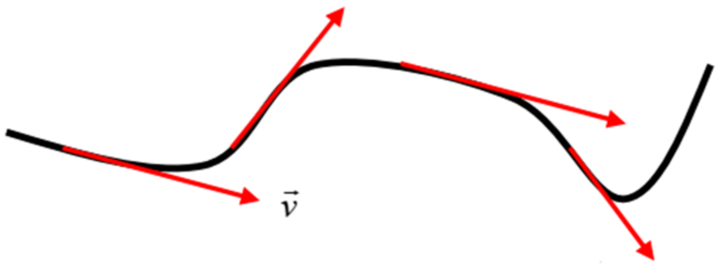
$$\lim_{\Delta t \to 0} \frac{\overrightarrow{OM}(t + \Delta t) - \overrightarrow{OM}(t)}{\Delta t}$$

$$\vec{v}_t = \lim_{t \to t'} \frac{\overrightarrow{OM'} - \overrightarrow{OM}}{t - t'} = \lim_{t' \to t} \frac{\Delta \overrightarrow{OM}}{\Delta t} = \frac{d\overrightarrow{OM}}{dt} \quad \boxed{\vec{v}_t = \frac{d\overrightarrow{OM}}{dt}}$$



In a Cartesian coordinate system, for example, we derive the expression of the instantaneous velocity vector from the expression of the position vector by taking the derivative.

$$\boxed{\overrightarrow{OM} = \vec{r} = x.\vec{i} + y.\vec{j} + z.\vec{k} \Rightarrow \vec{v} = \dot{x}.\vec{i} + \dot{y}.\vec{j} + \dot{z}.\vec{k}}$$



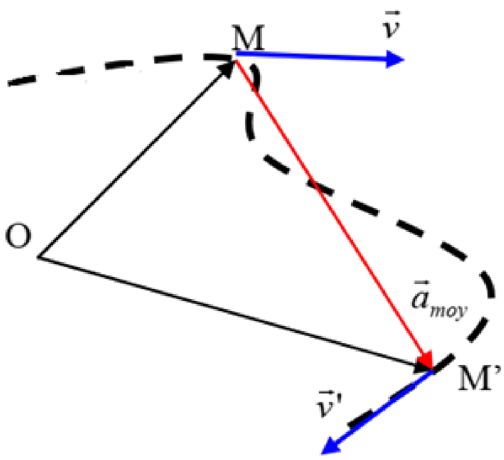
c) Magnitude of the Instantaneous Velocity Vector

$$\boxed{v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}$$

The unit of velocity in the International System is m/s (meters per second).

So, the components of the vectors $\overrightarrow{OM}$ and $\vec{v}$ in Cartesian coordinates are:

$$\overrightarrow{OM}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_R \rightarrow \vec{v}\begin{pmatrix} \dot{x} = v_x \\ \dot{y} = v_y \\ \dot{z} = v_z \end{pmatrix}_R$$



2.4. *THE ACCELERATION VECTOR*

We consider acceleration to be the change in velocity per unit of time.

a) Average Acceleration Vector

By considering two different moments, t and t', corresponding to the position vectors $\overrightarrow{OM}$ and , as well as the instantaneous velocity vectors $\vec{v}$ and $\vec{v}'$, the average acceleration vector is defined by the expression.

$$\lim_{\Delta t \to 0} \frac{\vec{V}(t+\Delta t) - \vec{V}(t)}{\Delta t} = \frac{\Delta \vec{V}}{\Delta t} = \frac{dV}{dt}$$

$$\boxed{\vec{a}_{moy} = \frac{\vec{v}' - \vec{v}}{t' - t} = \frac{\Delta \vec{v}}{\Delta t}; \quad a_{moy} = \frac{|\Delta \vec{v}|}{\Delta t}}$$

b) Instantaneous Acceleration Vector

The instantaneous acceleration vector of a motion is defined as the derivative of the instantaneous velocity vector with respect to time.

$$\vec{a} = \lim_{t' \to t} \frac{\vec{v'} - \vec{v}}{t' - t} = \lim_{t' - t} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2 \overrightarrow{OM}}{dt^2} \quad \boxed{\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2 \overrightarrow{OM}}{dt^2}}$$

$$\overrightarrow{OM} = \vec{r} = x.\vec{i} + y.\vec{j} + z.\vec{k} \Rightarrow \vec{v} = \dot{x}.\vec{i} + \dot{y}.\vec{j} + \dot{z}.\vec{k} \quad \Rightarrow \vec{a} = \ddot{x}.\vec{i} + \ddot{y}.\vec{j} + \ddot{z}.\vec{k}$$

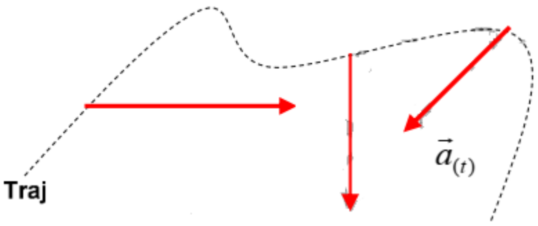
$$\vec{v} = \frac{dx}{dt}.\vec{i} + \frac{dy}{dt}.\vec{j} + \frac{dz}{dt}.\vec{k} \quad \Rightarrow \boxed{\vec{a} = \frac{d^2x}{dt^2}.\vec{i} + \frac{d^2y}{dt^2}.\vec{j} + \frac{d^2z}{dt^2}.\vec{k}}$$

" ***Important:*** *The acceleration vector is always directed towards the concave part of the trajectory.* "

c) Magnitude of the Instantaneous Acceleration Vector

This magnitude is given by the formula.

$$\boxed{a = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}}$$



" ***CONCLUSION:*** *In a Cartesian coordinate system, the position, velocity, and acceleration vectors are:* "

$$\begin{cases} a_x = \dfrac{dv_x}{dt} = \dfrac{d^2x}{dt^2} \\ a_y = \dfrac{dv_y}{dt} = \dfrac{d^2y}{dt^2} \\ a_z = \dfrac{dv_z}{dt} = \dfrac{d^2z}{dt^2} \end{cases}$$

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_R \rightarrow \vec{v} \begin{pmatrix} \dot{x} = v_x \\ \dot{y} = v_y \\ \dot{z} = v_z \end{pmatrix}_R \rightarrow \vec{a} \begin{pmatrix} \ddot{x} = \dot{v}_x = a_x \\ \ddot{y} = \dot{v}_y = a_y \\ \ddot{z} = \dot{v}_z = a_z \end{pmatrix}_R$$

$$\overrightarrow{OM} = \vec{r} = x.\vec{i} + y.\vec{j} + z.\vec{k} \rightarrow \vec{v} = v_x.\vec{i} + v_y.\vec{j} + v_z.\vec{k} \rightarrow \vec{a} = a_x.\vec{i} + a_y.\vec{j} + a_z.\vec{k}$$

" ***Note:*** *The motion is considered accelerated if* $\vec{a} \bullet \vec{v} > 0$ *and decelerated or retarded if* $\vec{a} \bullet \vec{v} < 0$*. The direction of the motion is indicated by the direction of the velocity vector* $\vec{v}$*.* "

d) Example

Example

Let's consider the position vector $\overrightarrow{OM}$

$$\overrightarrow{OM} \begin{pmatrix} x = 2t^2 \\ y = 4t - 5 \\ z = t^3 \end{pmatrix}$$

From this, deduce the instantaneous velocity vector and the instantaneous acceleration vector, then calculate the magnitude of each.

Answer:

We differentiate the expression of the position vector twice to obtain the requested vectors and then deduce their magnitudes.

$$\vec{v} = 4t.\vec{i} + 4\vec{j} + 3t^2\vec{k} \quad \rightarrow \quad \vec{a} = 4\vec{i} + 0\vec{j} + 6t\vec{k}$$

$$v = \sqrt{16t^2 + 16 + 9t^4} \quad , \quad a = \sqrt{16 + 36t^2}$$

3. *RECTILINEAR MOTION*

3.1. UNIFORM RECTILINEAR MOTION

Az Definition

A material point is in uniform rectilinear motion if its trajectory is a straight line, and its velocity vector is constant (thus, its acceleration vector is zero).

a) Time Equation

We choose the ***OX*** axis as a rectilinear reference frame and establish the initial condition **t = 0**; **x =** x_0 (initial abscissa).

Starting from the definition above and through integration, we can express the abscissa ***x*** as a function of time.

$$v = \dot{x} = \frac{dx}{dt} = v_0 \Rightarrow dx = v_0.dt \Rightarrow \int_{x_0}^{x} dx = \int_{t_0}^{t} v_0.dt$$

$$x\Big|_{x_0}^{x} = v_0 t\Big|_0^t \quad \Rightarrow x\text{-}x_0 = v_0 t$$

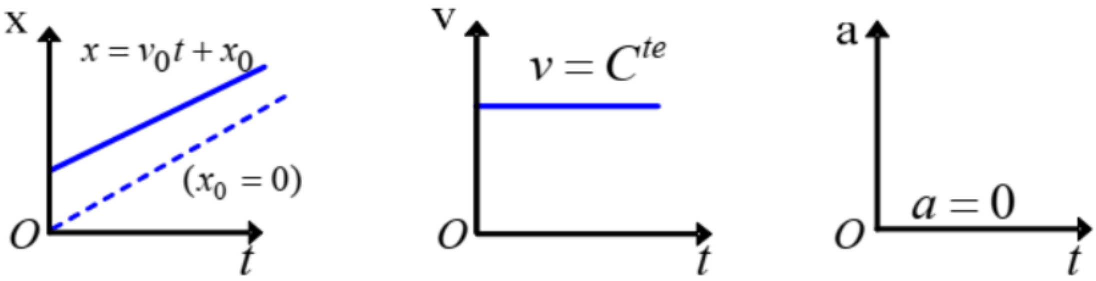
In a final step, we obtain the time equation of rectilinear motion, which is a first-degree function of time.

$$\boldsymbol{x = v_0\, t + x_0}$$

We call $\boldsymbol{x}$ the instantaneous abscissa, and $\boldsymbol{x_0}$ the initial abscissa.

b) Motion Diagrams

Motion diagrams for uniform rectilinear motion are graphical representations of acceleration, velocity, and displacement as functions of time.



c) Example

Example

The time equations of motion for a material point are given as **x(t) = 2t**, **y(t) = 2t + 4**, and **z(t) = 0** (all units are in the International System). Show that the motion is rectilinear and uniform.

" ***Note:*** *In a Cartesian coordinate system, if one of the coordinates is zero, the motion is called planar (but it can also be rectilinear); if two coordinates are zero, the motion can only be rectilinear; if all three coordinates are nonzero, then the motion is called spatial.* "

Answer: First, let's demonstrate that the motion is rectilinear. To do that, we need to find the equation of the trajectory. After eliminating time between the two given time equations, we find **y = x + 4**, which is the equation of a straight line, indicating that the motion is rectilinear. For this motion to be uniform, the velocity must be constant in direction, sense, and magnitude. The velocity vector is

$$\vec{v} = 2\vec{i} + 2\vec{j} \Rightarrow \ v = \sqrt{2^2 + 2^2} \Rightarrow v = \sqrt{8} = 2.83ms^{-1}$$

This implies that the motion is uniform. In conclusion, the motion is rectilinear and uniform.

3.2. RECTILINEAR UNIFORMLY VARIED MOTION

Az Definition

The motion of a material point is rectilinear uniformly varied if its trajectory is a straight line and its acceleration is constant.

a) Algebraic velocity

Considering the initial conditions $t = 0$ and $v = v_0$ (initial velocity), and based on the previous definitions, through integration, we can write:

$$a = \frac{dv}{dt} \Rightarrow dv = adt \cdot \int_{v_0}^{v} dv = \int_{t_0}^{t} adt \Rightarrow v\Big|_{v_0}^{v} = at\Big|_{0}^{t}$$

At the end, we obtain the equation for instantaneous velocity as a first-degree function of time.

$$v = \frac{dx}{dt} = a_0(t - t_0) + v_0$$

b) Equation of motion

If we take $t_0 = 0$ and $x = x_0$ (initial position), based on the above, we write:

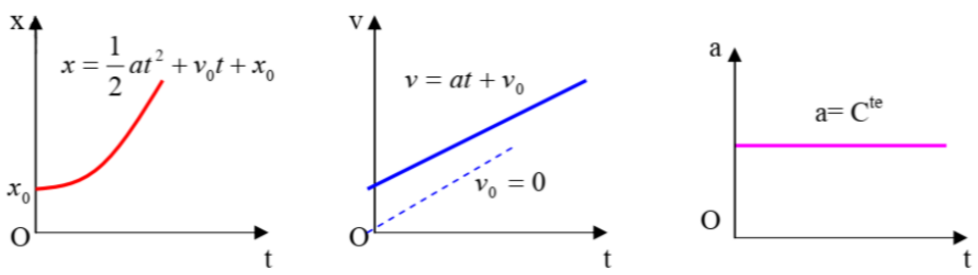
$$v = \frac{dx}{dt} = a_0(t - t_0) + v_0 \quad \int_{x_0}^{x} dx = \int_{t_0}^{t} (a_0(t - t_0) + v_0)dt \Rightarrow$$

The equation of motion is thus:

$$x(t) = \frac{1}{2}a_0(t - t_0)^2 + v_0(t - t_0) + x_0$$

c) Motion Diagrams

In next Figure, you can see the diagrams for rectilinear uniformly varied motion related to acceleration, velocity, and displacement.



d) Example

Example

A point particle moves along the **X-axis** with a velocity given by:

$$v = 2t - 6 \quad (ms^{-1})$$

a/ From this, deduce the equation for acceleration and the time equation for this motion, knowing that at the moment **t = 0, x = 5 m**. What is the nature of the motion?

b/ Specify the stages (accelerated and decelerated) of the motion.

Answer: We obtain the equation for acceleration by differentiating the expression for velocity with respect to time:

$$a = \frac{dv}{dt} = 2ms^{-2}$$

a. The acceleration is constant. By integrating the expression for velocity, we obtain the time equation:

$$v = \frac{dx}{dt} \Rightarrow x = x_0 + \int_0^t v dt \Rightarrow x = x_0 + \int_0^t (2t-6) \Big| \Rightarrow \boxed{x = t^2 - 6t + 5}$$

$$x = x_0 + t^2 - 6t \;;\; t = 0 \;,\; x = 5 \Rightarrow x_0 = 5$$

The motion is rectilinear uniformly varied.

b/ Phases of the motion: We create the following table of variations:

t	0	1	3	5	∞
v		-	0	+	
a		+		+	
x	↘	0	-4	0	↗
av		-	0	+	

Retarded Motion Accelerated Motion

3.3. RECTILINEAR MOTION WITH VARIABLE ACCELERATION

Definition

The motion of a material point is called rectilinear with variable acceleration if its trajectory is a straight line, and its acceleration is a function of time **(a = f(t)).**

a) Example

Example

A point particle moves along a straight line with an acceleration given by **a = 4 - t^2** Find the expressions for velocity and displacement as functions of time, considering the following conditions: **t = 3** seconds, **v = 2 m/s, x = 9** meters.

Answer: To obtain the symbolic expression for velocity, we need to integrate the acceleration equation:

$$v = \int_0^t a dt + v_0 \Rightarrow v = v_0 + \int_0^t (4 - t^2) dt \quad v = 4t - \frac{1}{3}t^3 + v_0$$

Integrating again to obtain the symbolic expression for displacement:

$$x = x_0 + \int_0^t v dt \quad \Rightarrow x = -\frac{1}{12}t^4 + 2t^2 - v_0 t + x_0$$

We now need to determine the initial position and initial velocity of the object. Based on the given data, we substitute **t = 3** seconds into the previously obtained expressions to find the initial position and initial velocity:

$$t = 3s \Rightarrow x_0 = \frac{3}{4}m \quad ; \quad v_0 = -1ms^{-1}$$

In the end, the expressions for velocity and displacement are as follows:

$$\boxed{x = 2t^2 - \frac{1}{12}t^4 - t + \frac{3}{4}}$$

$$\boxed{v = 4t - \frac{1}{3}t^3 - 1}$$

3.4. RECTILINEAR SINUSOIDAL MOTION

Az Definition

The motion of a material point is called rectilinear sinusoidal if its equation of motion can be written in the form:

$$\boxed{x = X_m.\cos(\omega.t + \varphi)} \quad \boxed{x = X_m.\sin(\omega.t + \varphi)}$$

Where:

- X_m Amplitude or maximum elongation, measured in meters.
- x: Instantaneous elongation or position, varying between two extreme values: $-1 \leq \cos(\omega t + \varphi) \leq +1 \Rightarrow -X_m \leq x \leq +X_m$

Measured in meters.

- **ω:** Angular frequency of the motion, measured in radians per second.
- **φ:** Initial phase, measured in radians.
- **ωt + φ:** Instantaneous phase, measured in radians.

a) Velocity

By differentiating the equation of motion, we obtain the expression for instantaneous velocity:

$$v = \dot{x} = \frac{dx}{dt} \quad \boxed{v = -X_m.\omega\sin(\omega t + \varphi)}$$

This velocity varies between two extreme values: $-1 \leq \sin(\omega t + \varphi) \leq +1 \Rightarrow -X_m.\omega \leq v \leq +X_m.\omega$

b) Acceleration

By differentiating the equation of velocity, we obtain the expression for instantaneous acceleration:

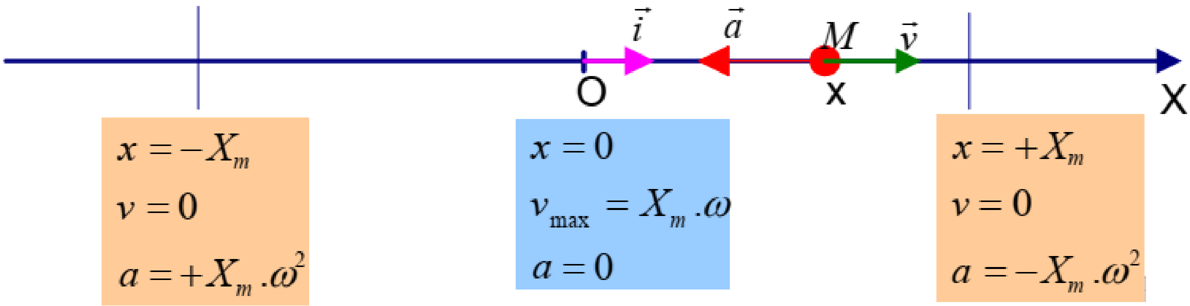
$$a = \ddot{x} = \dot{v} = \frac{dv}{dt} \quad \boxed{a = -X_m\omega^2\cos(\omega.t + \varphi)}$$

This acceleration varies between two extreme values $+X_m\omega^2 \geq a \geq -X_m\omega^2$

We can express the expression of acceleration in the form: $\boxed{a = -\omega^2 .x}$

Acceleration is proportional to the elongation with an opposite sign.

Unlike velocity, acceleration becomes zero when the object passes through the equilibrium position (origin of the abscissas), and it reaches its maximum value when elongation is at its maximum. The main characteristics of rectilinear sinusoidal motion are summarized in next Figure



c) Differential Equation of Motion

The equation of acceleration can be written in the form of a differential equation:

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 .x = 0} \quad a = \ddot{x} = -\omega^2 x \Rightarrow \ddot{x} + \omega^2 .x = 0$$

The mathematical solution to this differential equation is of the form:

$$x = A \cos \omega t + B \sin \omega t$$

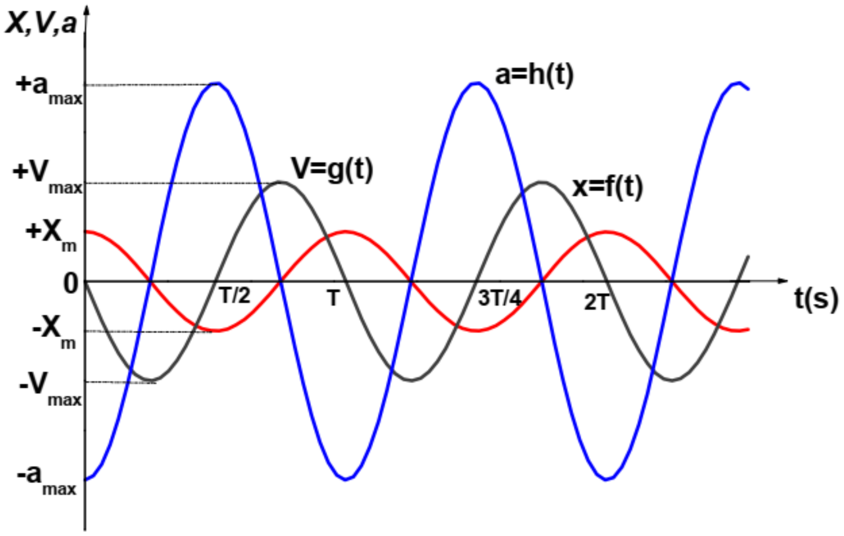
After trigonometric transformation, we can write: $x = X_m \cos(\omega t + \varphi)$

Where and φ are the differential constants determined by the initial conditions on elongation and velocity . This leads to a system of two equations with two unknowns that allows us to determine and φ.

$$t = 0 \rightarrow \left| \begin{array}{l} x_0 = X_m \cos\varphi \\ v_0 = -X_m \sin\varphi \end{array} \right.$$

d) Motion Diagrams

Next Figure illustrates the diagrams of displacement, velocity, and acceleration for rectilinear sinusoidal motion (for simplicity, we have chosen $\varphi=0$).



e) Example

Example

A sinusoidal vibrator represented by the equation ***x = 4 sin (0.1t + 0.5)***.

Find:

a/ The amplitude, period, frequency, and initial phase of the motion.

b/ The velocity and acceleration.

c/ The initial conditions.

d/ The position, velocity, and acceleration at time ***t = 5*** seconds.

e/ Draw the motion diagrams.

Answer:

Let's proceed by identifying the general equation for rectilinear sinusoidal motion and the equation given in the statement of this exercise.

$$x = 4\sin(0.1t + 0.5) \quad = X_m \sin(\omega t + \varphi)$$

a/ The amplitude, period, frequency, and initial phase of the motion.

$$X_m = 4m \quad T = \frac{2\pi}{\omega} \Rightarrow \boxed{T = 20\pi = 62.8s}$$

$$N = \frac{1}{T} \Rightarrow \boxed{N = 1.59.10^{-2}\,Hz}; \qquad \boxed{\varphi = 0.5rad}$$

b/ Calculation of velocity and acceleration:

$$\boxed{v = \dot{x} = 0.4\cos(0.1t + 0.5)} \quad a = \dot{v} = -0.04\sin(0.1t + 0.5) = -0.04x \quad \boxed{a = -0.04x}$$

c/ Determination of initial conditions:

$$t = 0 \Rightarrow x_0 = 4\sin 0.5 = 1.92m \Rightarrow \boxed{x_0 = 1.92m}$$

$$v_0 = 0.4\cos 0.5 \approx 0.35ms^{-1} \Rightarrow \boxed{v_0 = 0.35m}$$

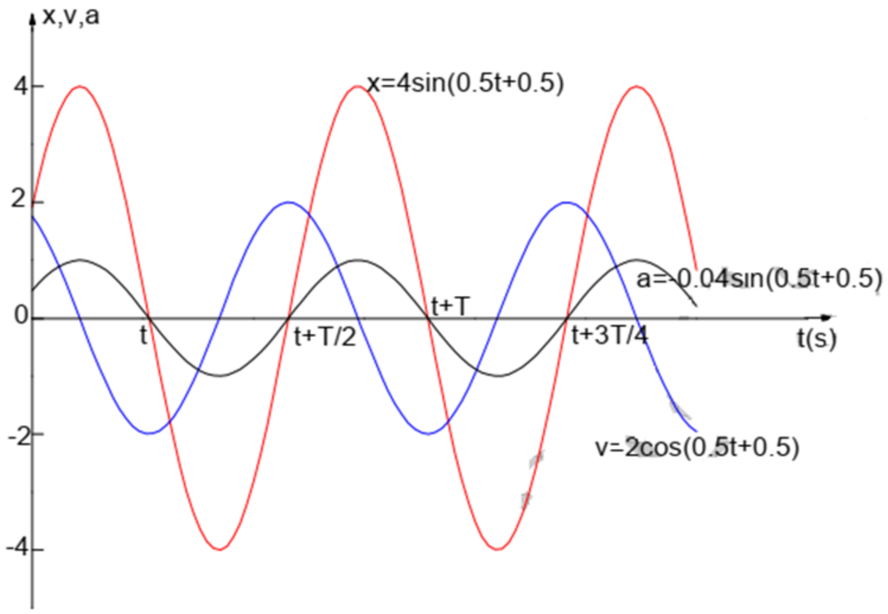
d/ Determination of position, velocity, and acceleration at time ***t=5*** seconds:

$$t = 5s: \; x = 4sin(0.5 + 0.5) \Rightarrow \boxed{x = 3.36m}$$

$$v = 0.4cos1 \Rightarrow \boxed{v = 0.22ms^{-1}}$$

$$a = -0.04sin1 \Rightarrow \boxed{a = 0.034ms^{-2}}$$

e/ Motion diagrams.



4. *MOTION IN THE PLANE*

If the trajectory lies in a plane, it is possible to locate the position of a mobile either using rectangular coordinates or polar coordinates.

4.1. *STUDY OF MOTION IN POLAR COORDINATES*

a) Position of the mobile

Let ***M*** be a material point whose trajectory is any plane curve **(C)**. The position of the mobile in Cartesian coordinates, as already noted, is defined as:

$$\boxed{\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j}}$$

But in polar coordinates, the position vector is written as:

$$\overrightarrow{OM} = \rho\, \vec{u}_\rho$$

Where: $\vec{u}_r = \vec{i}.\cos\theta + \vec{j}.\sin\theta$

So: $\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} = \rho\vec{u_\rho} = \rho\cos\theta\vec{i} + \rho\sin\theta\vec{j}$

" ***Note:*** *ρ and θ depend on time: ρ=* ***f(t)****, and θ=* ***g(t)****.* "

b) Velocity

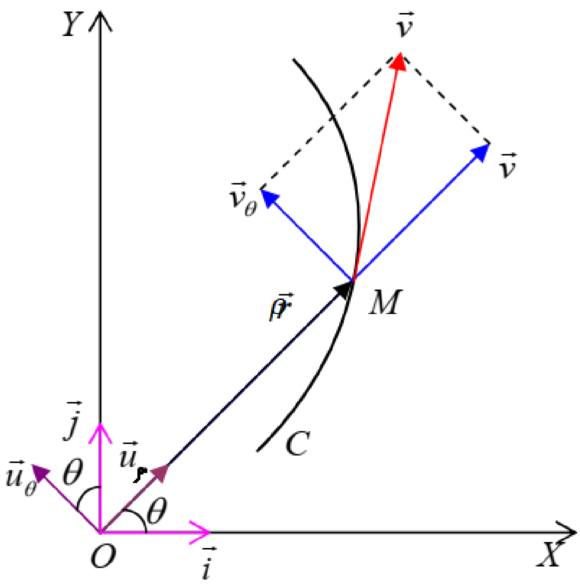
i) In Cartesian coordinates

$$\boxed{\vec{v} = \dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j}}$$

ii) In polar coordinates

According to next Figure, we can express the expressions of the two unit vectors $\vec{u_\rho}$ and $\vec{u_\theta}$ in terms of the unit vectors $\vec{i}$ and $\vec{j}$:

$$\begin{cases} \vec{u_\rho} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \\ \vec{u_\theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} \end{cases}$$



Their consecutive derivatives are:

$$\frac{\overrightarrow{du_\rho}}{dt} = -\sin\theta\frac{d\theta}{dt}\vec{i} + \cos\theta\frac{d\theta}{dt}\vec{j} = \vec{u}_\theta\frac{d\theta}{dt} \qquad \boxed{\frac{\overrightarrow{du_\rho}}{dt} = \vec{u}_\theta\frac{d\theta}{dt}}$$

$$\frac{\overrightarrow{du_\theta}}{dt} = -\cos\theta\frac{d\theta}{dt}\vec{i} - \sin\theta\frac{d\theta}{dt}\vec{j} = -\vec{u}_\rho\frac{d\theta}{dt} \qquad \boxed{\frac{\overrightarrow{du_\theta}}{dt} = -\vec{u}_\rho\frac{d\theta}{dt}}$$

Using these relations, let's express velocity in polar coordinates:

$$\vec{v} = \dot{\vec{\rho}} = \rho\frac{\overrightarrow{du_\rho}}{dt} + \vec{u}_\rho\frac{d\rho}{dt} \Rightarrow \vec{v} = \frac{d\rho}{dt}\vec{u}_\rho + \rho\frac{d\theta}{dt}\vec{u}_\theta \Rightarrow \boxed{\vec{v} = \dot{\rho}\,\vec{u}_\rho + \rho\,\dot{\theta}\,\vec{u}_\theta}$$

As a result, velocity has two components, transverse velocity $\vec{v}_\theta$ and radial velocity $\vec{v}_\rho$. Below are the expressions for both components and the magnitude of velocity in polar coordinates:

$$\begin{matrix} \vec{v} = \dot{\rho}\,\vec{u}_\rho + \rho\,\dot{\theta}\,\vec{u}_\theta \\ \vec{v} = \vec{v}_\rho + \vec{v}_\theta \end{matrix} \Rightarrow \begin{matrix} \vec{v}_\rho = \dot{\rho}\,\vec{u}_\rho \\ \vec{v}_\theta = \rho\,\dot{\theta}\,\vec{u}_\theta \end{matrix} \Rightarrow v = \sqrt{\dot{\rho}^2 + (\rho\,\dot{\theta})^2}$$

c) Acceleration

i) In Cartesian coordinates

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{\rho}} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath}$$

ii) In polar coordinates

We differentiate the velocity relation, with respect to time, we obtain the formula for acceleration.

$$\vec{a} = \dot{\vec{v}} = \dot{\rho}\,\frac{\overrightarrow{du_\rho}}{dt} + \ddot{\rho}\,\vec{u}_\rho + \rho\,\dot{\theta}\,\frac{\overrightarrow{du_\theta}}{dt} + \rho\,\ddot{\theta}\,\vec{u}_\theta + \dot{\rho}\,\dot{\theta}\,\vec{u}_\theta$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{\rho}\,\overrightarrow{u_\rho} + \rho\dot{\theta}\overrightarrow{u_\theta}\right)$$

$$=> \vec{a}(t) = \ddot{\rho}\overrightarrow{u_\rho} + \dot{\rho}\dot{\overrightarrow{u_\rho}} + \dot{\rho}\dot{\theta}\overrightarrow{u_\theta} + \rho\ddot{\theta}\overrightarrow{u_\theta} + \rho\dot{\theta}\dot{\overrightarrow{u_\theta}}$$

$$=> \vec{a}(t) = \ddot{\rho}\overrightarrow{u_\rho} + \dot{\rho}\dot{\theta}\overrightarrow{u_\theta} + \dot{\rho}\dot{\theta}\overrightarrow{u_\theta} + \rho\ddot{\theta}\overrightarrow{u_\theta} + \rho\dot{\theta}(-\dot{\theta}\,\overrightarrow{u_\rho})$$

$$\vec{a}(t) = (\ddot{\rho} - \rho\dot{\theta}^2)\overrightarrow{u_\rho} + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\overrightarrow{u_\theta} \qquad \vec{a}(t) = a_\rho\,\overrightarrow{u_\rho} + a_\theta\overrightarrow{u_\theta}$$

$$\begin{cases} a_\rho = \ddot{\rho} - \rho\,\dot{\theta}^2 \\ a_\theta = 2\,\dot{\rho}\dot{\theta} + \rho\,\ddot{\theta} \end{cases} \Longrightarrow \|\vec{a}(t)\| = \sqrt{a_\rho^2 + a_\theta^2} = \sqrt{(\ddot{\rho} - \rho\,\dot{\theta}^2)^2 + (2\,\dot{\rho}\dot{\theta} + \rho\,\ddot{\theta})^2}$$

d) Special case, Circular motion

Since ρ ***= R = Cte***, the velocity vector is therefore:

$$\vec{v} = \rho\,\dot{\theta}\,\vec{u}_\theta$$

And the expression for the acceleration vector is:

$$\vec{a} = -\rho\,\dot{\theta}^2\,\vec{u}_\rho + \rho\,\ddot{\theta}\,\vec{u}_\theta$$

Notice that this acceleration has two components:

Normal acceleration, denoted by $\vec{a}_N$, carried by the normal, directed towards the center, and opposite in direction to $\vec{a}$. It indicates the change in the direction of velocity.

$$\vec{a}_N = -\vec{a}_\rho = \rho\,\dot{\theta}^2\,\vec{u}_\rho \Longrightarrow a_\rho = a_N = \rho\,\dot{\theta}^2$$

Tangential acceleration, denoted by $\vec{a}_T$, carried by the tangent to the trajectory at point ***M***, indicates the change in the magnitude of velocity.

$$\vec{a}_T = \vec{a}_\theta = \rho\,\ddot{\theta}\,\vec{u}_\theta \Longrightarrow a_\theta = a_T = \rho\,\ddot{\theta}$$

5. *MOTION IN THE SPACE*

To study the motion of a material point in space characterized by three dimensions, cylindrical and spherical coordinates are generally used.

5.1. *STUDY OF MOTION IN CYLINDRICAL COORDINATES*

" Cylindrical coordinates = polar coordinates + z "

a) Position of the mobile

The position vector is given by: $\overrightarrow{OM} = \rho\, \vec{u}_\rho + z\, \vec{k} \quad => \|\overrightarrow{OM}\| = \sqrt{\rho^2 + z^2}$

b) Velocity

According to the definition:

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt} = \frac{d(\rho\overrightarrow{u_\rho}+z\vec{k})}{dt} = \dot{\rho}\,\overrightarrow{u_\rho} + \rho\frac{d\overrightarrow{u_\rho}}{dt} + \dot{z}\,\vec{k} \qquad \text{or} \qquad \frac{d\overrightarrow{u_\rho}}{dt} = \dot{\theta}\overrightarrow{u_\theta}$$

$$\vec{v}(t) = \dot{\rho}\,\overrightarrow{u_\rho} + \rho\dot{\theta}\overrightarrow{u_\theta} + \dot{z}\,\vec{k} \qquad \vec{v}(t) = v_\rho\,\overrightarrow{u_\rho} + v_\theta\overrightarrow{u_\theta} + v_z\vec{k}$$

$$\begin{cases} v_\rho = \dot{\rho} \\ v_\theta = \rho\,\dot{\theta} \\ v_z = \dot{z} \end{cases} \Longrightarrow \|\vec{v}(t)\| = \sqrt{v_\rho^2 + v_\theta^2 + v_z^2} = = \sqrt{(\dot{\rho}^2 + \left(\rho\,\dot{\theta}\right)^2 + \dot{z}^2}$$

c) Acceleration

According to the definition:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{\rho}\,\overrightarrow{u_\rho} + \rho\dot{\theta}\overrightarrow{u_\theta} + \dot{z}\,\vec{k}\,\right)$$

$$=> \vec{a}(t) = \ddot{\rho}\overrightarrow{u_\rho} + \dot{\rho}\dot{\overrightarrow{u_\rho}} + \dot{\rho}\dot{\theta}\overrightarrow{u_\theta} + \rho\ddot{\theta}\overrightarrow{u_\theta} + \rho\dot{\theta}\dot{\overrightarrow{u_\theta}} + \ddot{z}\,\vec{k}$$

$$=> \vec{a}(t) = \ddot{\rho}\overrightarrow{u_\rho} + \dot{\rho}\dot{\theta}\overrightarrow{u_\theta} + \dot{\rho}\dot{\theta}\overrightarrow{u_\theta} + \rho\ddot{\theta}\overrightarrow{u_\theta} + \rho\dot{\theta}(-\dot{\theta}\,\overrightarrow{u_\rho}\,) + \ddot{z}\,\vec{k}$$

$$\vec{a}(t) = (\ddot{\rho} - \rho\dot{\theta}^2)\overrightarrow{u_\rho} + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\,\right)\overrightarrow{u_\theta} + \ddot{z}\,\vec{k}$$

$$\vec{a}(t) = a_\rho\,\overrightarrow{u_\rho} + a_\theta\overrightarrow{u_\theta} + a_z\vec{k}$$

$$\begin{cases} a_\rho = \ddot{\rho} - \rho\,\dot{\theta}^2 \\ a_\theta = 2\,\dot{\rho}\dot{\theta} + \rho\,\ddot{\theta} \\ a_z = \ddot{z} \end{cases} \Longrightarrow \|\vec{a}(t)\| = \sqrt{a_\rho^2 + a_\theta^2 + a_z^2} = \sqrt{(\ddot{\rho} - \rho\,\dot{\theta}^2)^2 + \left(2\,\dot{\rho}\dot{\theta} + \rho\,\ddot{\theta}\right)^2 + \ddot{z}^2}$$

5.2. *STUDY OF MOTION IN SPHERICAL COORDINATES*

a) Position of the mobile

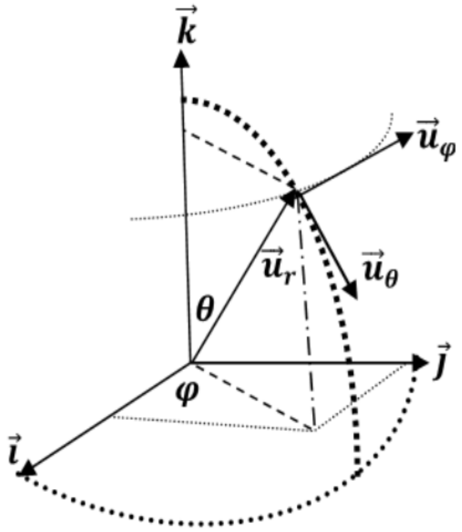
The position vector is given by: $\overrightarrow{OM} = r\,\vec{u}_r \quad => \quad \|\overrightarrow{OM}\| = r$

b) Velocity

According to the definition:

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt} = \dot{r}\ \overrightarrow{u_r} + r\dot{\overrightarrow{u_r}}$$

Spherical basis: $\overrightarrow{u_r}, \overrightarrow{u_\theta}, \overrightarrow{u_\varphi}$



$$\begin{cases} \overrightarrow{u_r} = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k} \\ \overrightarrow{u_\theta} = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k} \\ \overrightarrow{u_\varphi} = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{k} \end{cases}$$

$$\begin{cases} \dot{\overrightarrow{u_r}} = \dot{\theta}\,\overrightarrow{u_\theta} + \dot{\varphi}\sin\theta\,\overrightarrow{u_\varphi} \\ \dot{\overrightarrow{u_\theta}} = -\dot{\theta}\,\overrightarrow{u_r} + \dot{\varphi}\cos\theta\,\overrightarrow{u_\varphi} \\ \dot{\overrightarrow{u_\varphi}} = -\dot{\varphi}\,(\sin\theta\,\overrightarrow{u_r} + \cos\theta\,\overrightarrow{u_\theta}) \end{cases}$$

$$\vec{v}(t) = \dot{r}\ \overrightarrow{u_r} + r\dot{\overrightarrow{u_r}}$$

$$\Rightarrow\ \vec{v}(t) = \dot{r}\ \overrightarrow{u_r} + r\dot{\theta}\,\overrightarrow{u_\theta} + r\dot{\varphi}\sin\theta\,\overrightarrow{u_\varphi}$$

$$\vec{v}(t) = v_r\,\overrightarrow{u_r} + v_\theta\overrightarrow{u_\theta} + v_\varphi\overrightarrow{u_\varphi}$$

$$\begin{cases} v_\rho = \dot{r} \\ v_\theta = r\,\dot{\theta} \\ v_z = r\,\dot{\varphi}\,\sin\theta \end{cases} \Longrightarrow \|\vec{v}(t)\| = \sqrt{v_r^2 + v_\theta^2 + v_\varphi^2} = = \sqrt{\dot{r}^2 + (r\,\dot{\theta})^2 + (r\,\dot{\varphi}\,\sin\theta)^2}$$

c) Acceleration

According to the definition:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\ \overrightarrow{u_r} + r\dot{\theta}\,\overrightarrow{u_\theta} + r\dot{\varphi}\sin\theta\,\overrightarrow{u_\varphi}\right)$$

$$\vec{a}(t) = \ddot{r}\overrightarrow{u_r} + \dot{r}\dot{\overrightarrow{u_r}} + \dot{r}\dot{\theta}\,\overrightarrow{u_\theta} + r\ddot{\theta}\overrightarrow{u_\theta} + r\dot{\theta}\dot{\overrightarrow{u_\theta}} + \dot{r}\dot{\varphi}sin\theta\,\overrightarrow{u_\varphi}$$

$$+ r\ddot{\varphi}\sin\theta\,\overrightarrow{u_\varphi} + r\dot{\varphi}\dot{\theta}\cos\theta\,\overrightarrow{u_\varphi} + r\dot{\varphi}\sin\theta\,\dot{\overrightarrow{u_\varphi}}$$

$$\vec{a}(t) = \left(\ddot{r} - r\dot{\theta}^{\,2} - r\dot{\varphi}^2\,sin^2\,\theta\right)\overrightarrow{u_r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\ - r\dot{\varphi}\,sin\,\theta\,cos\,\theta\right)\overrightarrow{u_\theta}$$

$$+ \left(r\ddot{\varphi}\,sin\,\theta + 2r\dot{\varphi}\dot{\theta}\,cos\,\theta + 2\dot{r}\dot{\varphi}\,sin\,\theta\ \right)\overrightarrow{u_\varphi}$$

$$\vec{a}(t) = a_r\,\overrightarrow{u_r} + a_\theta\overrightarrow{u_\theta} + a_\varphi\overrightarrow{u_\varphi}$$

$$\begin{cases} a_r = \ddot{r} - r\,\dot{\theta}^2 - r\,\dot{\varphi}^2\,sin^2\theta \\ a_\theta = 2\,\dot{r}\dot{\theta} + r\,\ddot{\theta} - r\,\dot{\varphi}\,sin\,\theta\,cos\,\theta \\ a_\varphi = r\,\ddot{\varphi}\,sin\,\theta + 2\,r\,\dot{\varphi}\,\dot{\theta}\,cos\,\theta + 2\,r\,\dot{\varphi}\,\dot{s}in\,\theta \end{cases} \Longrightarrow \|\vec{a}(t)\| = \sqrt{a_r^2 + a_\theta^2 + a_\varphi^2}$$

$$\sqrt{(\ddot{r} - r\,\dot{\theta}^2 - r\,\dot{\varphi}^2\,sin^2\theta)^2 + \left(2\,\dot{r}\dot{\theta} + r\,\ddot{\theta} - r\,\dot{\varphi}\,sin\,\theta\,cos\,\theta\right)^2}$$

$$\overline{+\ (r\,\ddot{\varphi}\,sin\,\theta + 2\,r\,\dot{\varphi}\,\dot{\theta}\,cos\,\theta + 2\,r\,\dot{\varphi}\,\dot{s}in\,\theta)^2}$$

Conclusion

Chapter 1 delves into the kinematics of a material point, where we examined the nature of motion without considering the forces that cause it. We began with rectilinear motion, understanding how objects move along straight paths and in space. From there, we explored specific types of motion, analyzing the behavior of particles in different scenarios. Finally, we considered how movements are described in various coordinate systems, including polar, cylindrical, and spherical. This chapter has provided a comprehensive understanding of how to describe and analyze motion in its simplest form, setting the stage for exploring the causes of motion in the following chapters.