MECHANICS OF A MATERIAL POINT (Work and energy in the case of a material point)

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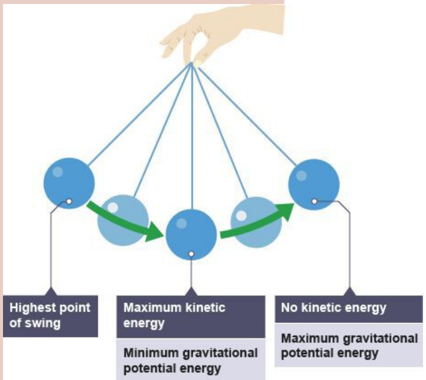

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Table of contents

I Chapter 3: WORK AND ENERGY

1. Introduction

In principle, Newton's laws allow solving all problems in classical mechanics. If one knows the initial positions and velocities of the particles in a system as well as all the forces acting on them, one can predict the evolution of the system over time.

However, in practice, we do not always know all the forces at play, and even if we do, the equations to solve may be too numerous or too complex. In many cases, valuable information about the system can be obtained more simply by invoking concepts such as work and energy.

2. *WORK AND POWER*

2.1. Power

Let ***M*** be a material point with velocity relative to the reference frame ***R***. The power of the force $\vec{F}$ acting on ***M*** is defined at each instant by the relation:

$$
\begin{array}{r|}
watt(W) \leftarrow P \\
N \leftarrow F \\
m.s^{-1} \leftarrow v
\end{array}
\quad \boxed{P = \vec{F}.\vec{v}}
$$

2.2. Work

The work done by force $\vec{F}$ between instant ***t***, when the material point ***M*** is at position $\overrightarrow{OM} = \vec{r}$, and instant ***t + dt***, when ***M*** is at position ***M'*** with $\overrightarrow{OM'} = \vec{r} + \overrightarrow{dr}$, is a quantity expressed in ***joules***:

$$\boxed{dW = P.dt}$$

According to the definition of velocity, we have:

$$\overrightarrow{OM'} - \overrightarrow{OM} = \overrightarrow{MM'} = d\vec{r} = \vec{v}.dt$$

Thus, we deduce the expression of the work done by force $\vec{F}$ for an elemental displacement $\overrightarrow{dr}$:

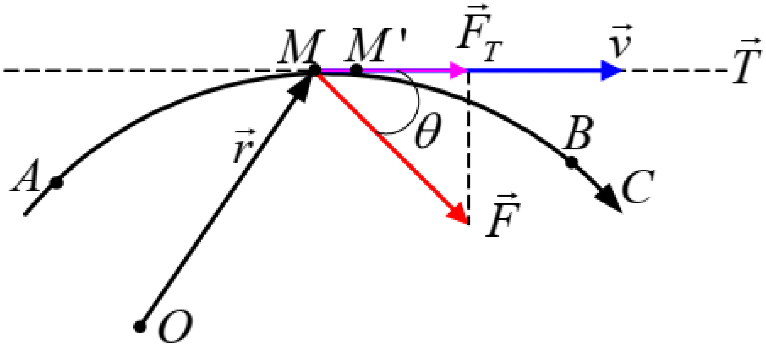
$$\boxed{dW = \vec{F}.d\vec{r}}$$

Note that work is a scalar product of the force vector and the displacement vector.

$$\boxed{dW = \|\vec{F}\|.\|d\vec{r}\|.\cos\theta}$$

Note that $F.\cos\theta = F_T$. If we set $\|d\vec{r}\| = ds$, we then obtain a new expression for work, which is:

$$\boxed{dW = F_T.ds}$$



For a total displacement from **A** (at time ***tA***) to **B** (at time ***tB***) along the curve **C**, we obtain the expression:

$$W = \int_A^B \vec{F}.d\vec{r} = \int_A^B F_T.ds$$

In the particular case where the force $\vec{F}$ has constant magnitude and direction, and the body moves along a straight trajectory, the work done by this force is:

$$F = F_T \Rightarrow W = \int_A^B F.ds = F\int_A^B ds \Leftrightarrow \boxed{W = F.s}$$

The force that does no work is the force perpendicular to the displacement ***($\theta = \pi/2$)***.

If Fx, Fy, Fz are the rectangular components of force $\vec{F}$, and ***dx, dy, dz*** are the rectangular components of the elemental displacement vector , then:

$$W = \int_A^B \vec{F}.d\vec{r} = \int_A^B (F_x.dx + F_y.dy + F_z.dz)$$

Case of multiple forces: If the body is subjected to several forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, ..., \vec{F}_n$, whose resultant is $\vec{F}_R$, the work done by these forces is equal to the work done by the resultant force:

$$dW = dW_1 + dW_2 + dW_3 + ... + dW_n$$

$$dW = \vec{F}_1.d\vec{r} + \vec{F}_2.d\vec{r} + \vec{F}_3.d\vec{r} + ... + \vec{F}_n.d\vec{r}$$

$$\boxed{dW = \vec{F}_R.d\vec{r}}$$

3. *KINETIC ENERGY*

We have seen that $dW = F_T ds$. Starting from this expression, we can deduce the following:

$$dW = F_T.ds = m\frac{dv}{dt}ds \Rightarrow dW = m\frac{ds}{dt}dv \Rightarrow \boxed{dW = mvdv}$$

Let's integrate the expression for elemental work and derive the definition of kinetic energy:

$$W = m\int_A^B vdv \Rightarrow \boxed{W = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2}$$

Where $\boldsymbol{v_A}$ is the velocity of the object at point ***A*** and $\boldsymbol{v_B}$ is its velocity at point ***B***.

The kinetic energy of a material point of mass ***m***, with velocity magnitude ***v***, is expressed as:

$$E_k = \frac{1}{2} m v^2$$

And since ***p = mv***, we can also write: $E_k = \frac{p^2}{2m}$

3.1. Kinetic energy theorem

Az Definition

"The change in kinetic energy of a material point between two instants is equal to the work done by the resultant of all forces applied to it between these two instants.:"

$$W = \Delta E_k \Leftrightarrow \sum_i W_i = \Delta E_k$$

(see video.mp4)

4. *CONSERVATIVE FORCES*

Az Definition

A force is termed conservative, or derived from a potential, if its work is independent of the path taken, regardless of the probable displacement between the starting point and the destination.

4.1. Example of weight

In a Cartesian coordinate system where **OZ** is the vertical axis oriented upwards, we have:

$$\vec{P} = \vec{F} = -mg\vec{k}$$

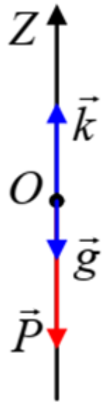
By using the expression of elemental displacement in Cartesian coordinates, we write:

$$d\vec{r} = dx.\vec{i} + dy.\vec{j} + dz.\vec{k}$$

From this, we can deduce: $dW = \vec{F}.d\vec{r} = -mgdz$ By integrating this latter expression, we realize that the work for a displacement between two points **A** and **B** doesn't depend on the path taken but only on their altitudes:

$$W = -\int_{z_1}^{z_2} mgdz \Rightarrow W = -mg(z_2 - z_1) \Rightarrow \boxed{W = mg(z_1 - z_2)}$$

If the two points are located in the same plane, the work done by weight is zero, proving that weight is a conservative force. $z_1 = z_2 \Rightarrow W = 0$



5. *POTENTIAL ENERGY*

Potential energy is a function of coordinates, such that the integration between its two values taken at the starting and ending points equals the work done on the particle to move it from its initial position to its final position.

If the force $\vec{F}$ is derived from a potential, then: $$W = \int_A^B \vec{F}.d\vec{r} = E_{p_A} - E_{p_B}$$

Potential energy is always referenced to a chosen origin in order to calculate it ***(Ep = 0)***. The potential energy function ***Ep*** is determined up to a constant.

The differential of potential energy is equal and opposite in direction to the differential of work.

$$dW = -dE_p(z) \quad \Leftrightarrow \quad dE_p(z) = -dW$$

5.1. Potential Energy of Some Force Fields

a) Particle in a uniform gravitational field

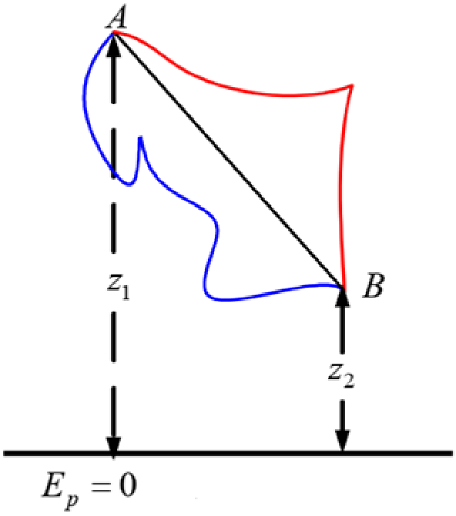
If **z** represents height measured from the Earth's surface, taken as the origin of potential energy, then the potential energy of the particle concerning the surface of the Earth is:

$$dE_p = -dW \Rightarrow \boxed{E_p = mgz}$$

In the general case, if the particle moves between two planes, the potential energy, regardless of the trajectory taken, is given by the formula:

$$E_p = mg(z_1 - z_2) \quad \begin{vmatrix} z_1 > z_2 \Rightarrow E_p > 0 \\ z_1 < z_2 \Rightarrow E_p < 0 \end{vmatrix}$$

More precisely, the calculated potential energy always represents the variation in its value between two given positions.



b) Particle Subjected to an Elastic Force

We will calculate the potential energy of a system composed of a particle attached to a spring, vertically suspended, with a stiffness constant **k**, and its length at rest being l_0. Its length when loaded with the particle is ***l***, therefore:

$$dE_p = -dW \quad ; \quad E_p = -\int_0^x -kxdx \Rightarrow \boxed{E_p = \tfrac{1}{2}k.x^2 = \tfrac{1}{2}k(l - l_0)^2}$$

6. *MECHANICAL ENERGY*

Az Definition

The mechanical energy of a material point at a given instant is equal to the sum of kinetic energy and potential energy

$$\boxed{E_M = E_k + E_p}$$

6.1. The principle of conservation of mechanical energy

In a conservative force field (or derived from a potential), mechanical energy remains constant over time.

$$E_M = E_k + E_p = C^{te}$$

This implies that the change in mechanical energy is zero $(\Delta E_M = 0)$, which also means that the change in kinetic energy is equal to the change in potential energy:

$$\Delta E_k = -\Delta E_p$$

In other words, if the system is mechanically isolated, mechanical energy is conserved. In the presence of friction, the change in mechanical energy equals the sum of the work done by frictional forces $W(\vec{F}_{fr})$

$$\boxed{\Delta E_M = \sum W_{fr}}$$

Conclusion

Chapter 3 focused on the interplay between work, energy, and forces in the context of a material point. We began by exploring kinetic energy and potential energies, such as gravitational and elastic, understanding how they are related to the motion of particles. The concept of a force field was introduced, allowing us to describe how forces vary across space. We also examined non-conservative forces, which do not conserve mechanical energy, contrasting them with conservative forces. This chapter has provided a nuanced understanding of how energy and work are related to the forces acting on a material point, enriching our grasp of the mechanical world.