Experiment 03. Coupled Pendulums

Experiment progression:

This report prepared by:

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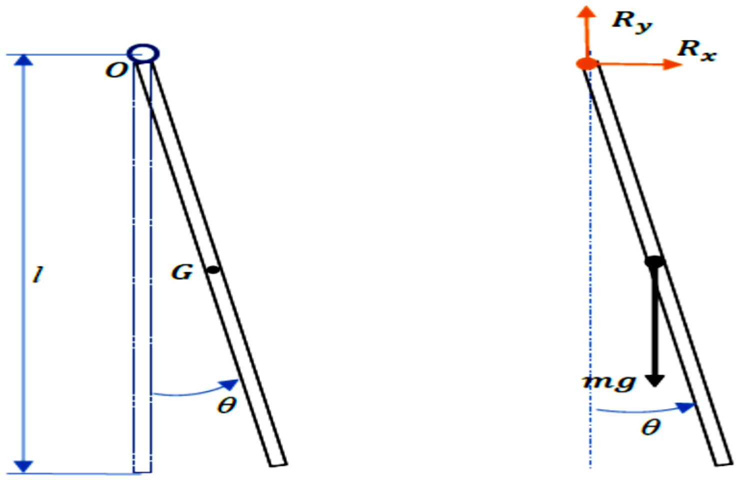
Objectives

- Verify experimentally that the movement of a system of two coupled pendulums can be considered as a superposition of two normal modes.

I. Physical pendulum

The pendulum consists of a rod of length *l* and mass m that can freely oscillate under the influence of its weight around an axis O. The distance between the center of mass of the pendulum and the axis of rotation is *l*/2.



Free body diagram of the physical pendulum.

The equation of movement.

We apply the second law of Newton of moments around the point O

$$\sum \mathcal{M}_{/O} = J_0\,\alpha$$

$$\overrightarrow{\mathcal{M}_{f_r/_O}} = \overrightarrow{OA} \wedge \overrightarrow{f_r}$$

$$\overrightarrow{(\mathcal{M}_{P/_O})} = \overrightarrow{OG} \wedge \overrightarrow{mg}$$

$$(\mathcal{M}_{P/_O}) = -\frac{l}{2} mg.\sin\theta$$

(The moments of the reaction forces R_x and R_y around point *O* are zero.).

$$J_O \frac{d^2\theta}{dt^2} = -mg\frac{l}{2}\sin\theta$$

For small vibration amplitudes, we use the approximation: sinθ≈θ, which gives:

$$J_O\,\ddot{\theta} + \frac{mgl}{2}\theta = 0$$

$$\omega_n = \sqrt{\frac{mgl}{2J_O}}$$

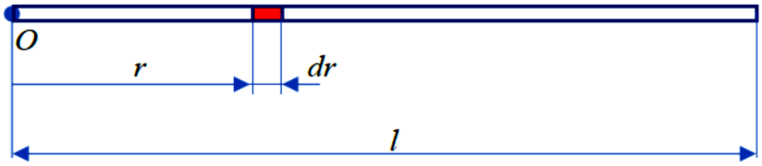
The natural frequency is: $$f_n = \frac{1}{2\pi}\sqrt{\frac{3}{2}\frac{g}{l}}$$

The period is $$T_n = 2\pi\sqrt{\frac{2}{3}\frac{l}{g}}$$

Moment of inertia.

The expression given for $J = ml^2$ is valid only for a pendulum formed by a point mass (mathematical pendulum). In the case where the mass has a certain spatial extent (physical pendulum), we need to consider its geometry. In our case, the pendulum consists of a prismatic rod of length l. Therefore, the center of mass is not located in the middle of the rod. Additionally, we must take into account the mass distribution when calculating the total moment of inertia of the pendulum.

$$J = \int_0^l r^2\, dm$$



Here, r is the distance between a mass element dm and the axis of rotation. $dm = \mu dr$; $\mu = m/l$ is the linear mass density.

$$J_O = \int_0^l r^2 \frac{m}{l} dr = \frac{1}{3}\frac{m}{l} r^3 |_0^l = \frac{1}{3} ml^2$$

To calculate the moment of inertia with respect to the center of mass of a rod:

$$J_G = \int_{-l/2}^{l/2} r^2 \frac{m}{l} dr = \frac{1}{3}\frac{m}{l} r^3 |_{-l/2}^{+l/2} = \frac{1}{3}\frac{m}{l}\left(\left(\frac{l}{2}\right)^3 - \left(-\frac{l}{2}\right)^3\right) = \frac{1}{12} ml^2$$

It is important to understand that the moment of inertia depends on the choice of the axis of rotation. In practice, it is often easier to calculate the moment of inertia JG with respect to an axis that passes through the center of mass G. To determine the moment of inertia with respect to another axis that is parallel to the first, the parallel axis theorem (Huygens' theorem) applies:

$$J_O = J_G + md^2$$

where m is the total mass of the rotating body and d is the distance between the center of mass and the axis of rotation.

For a rod with negligible cross-sectional dimensions compared to its length:

$$J_O = J_G + m\left(\frac{l}{2}\right)^2 = \frac{1}{12} ml^2 + \frac{1}{4} ml^2 = \frac{1}{3} ml^2$$

There are three fundamental types of oscillations:

The initial conditions:

$$\dot{\theta}_1(t) = \dot{\theta}_2(t) = 0$$
$$-\omega_1 A_1 \sin 0 + \omega_1 B_1 \cos 0 - \omega_2 A_2 \sin 0 + \omega_2 B_2 \cos 0 = 0$$
$$-\omega_1 A_1 \sin 0 + \omega_1 B_1 \cos 0 + \omega_2 A_2 \sin 0 - \omega_2 B_2 \cos 0 = 0$$
$$\omega_1 B_1 + \omega_2 B_2 = 0$$
$$\omega_1 B_1 - \omega_2 B_2 = 0$$
$$\Rightarrow B_1 = B_2 = 0$$

1. <u>Symmetric oscillations.</u>

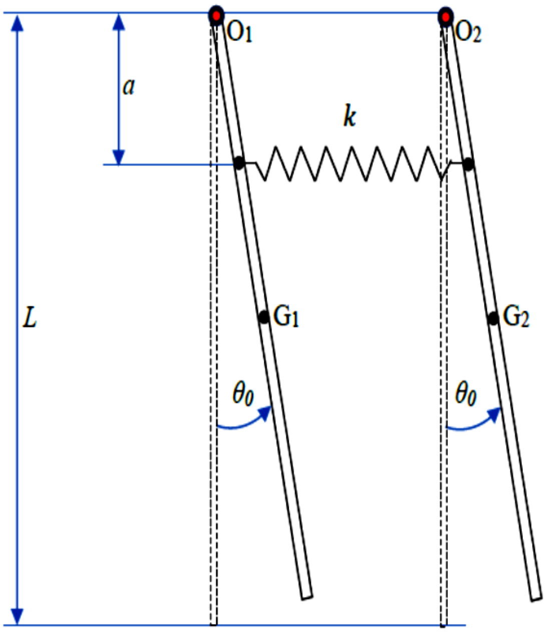
Initial conditions:

$$\theta_1(0) = \theta_2(0) = \theta_0$$
$$\dot{\theta}_1(t) = \dot{\theta}_2(t) = 0$$

$$A_1 \cos 0 + A_2 \cos 0 = \theta_0$$
$$A_1 \cos 0 - A_2 \cos 0 = \theta_0$$

$$\Rightarrow A_1 = \theta_0 \quad \text{et} \quad A_2 = 0$$

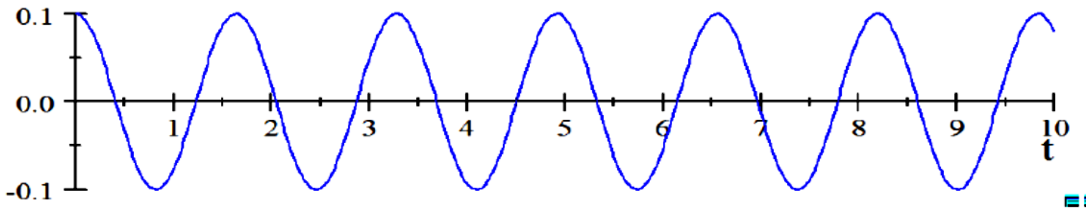
$$\theta_1(t) = \theta_0 \cos \omega_1 t$$
$$\theta_2(t) = \theta_0 \cos \omega_1 t$$



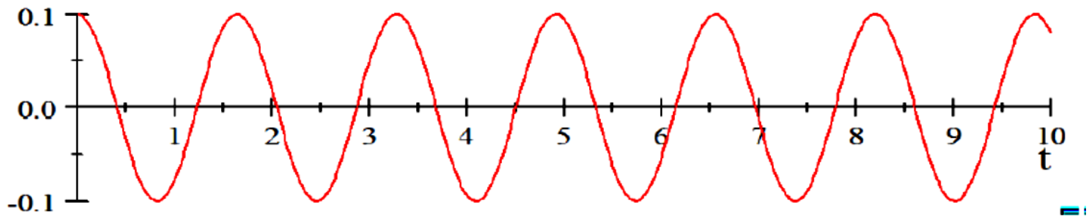
This is an oscillation at a single frequency. Coupling plays no role since ***the spring remains in the same state of tension***. Therefore, it is natural to find the period of the physical pendulum.

$$T_1 = 2\pi\sqrt{\frac{2l}{3g}}$$

$$\theta_1(t) = 0.1\cos(3.834t)$$



$$\theta_2(t) = 0.1\cos(3.834t)$$



2. Antisymmetric oscillations

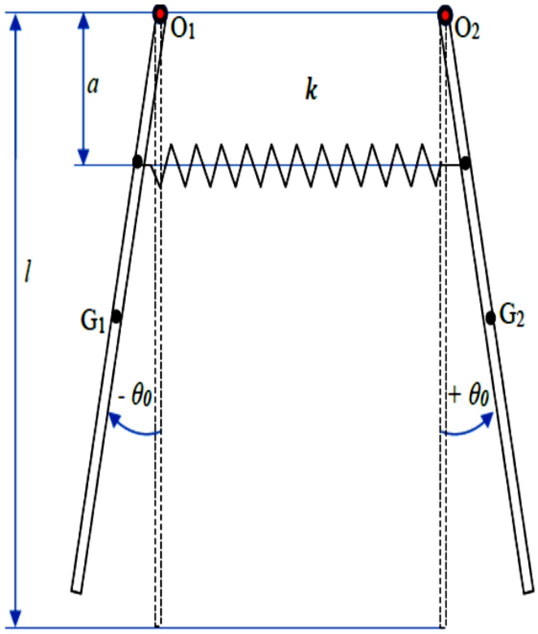
Initial conditions:

$$\theta_1(0) = -\theta_0 \quad \text{et} \quad \theta_2(0) = \theta_0$$
$$\dot{\theta}_1\ (t) = \dot{\theta}_2\ (t) = 0$$

$$A_1 \cos 0 + A_2 \cos 0 = -\theta_0$$
$$A_1 \cos 0 - A_2 \cos 0 = \theta_0$$

$$\Rightarrow A_1 = 0 \quad \text{et} \quad A_2 = -\theta_0$$

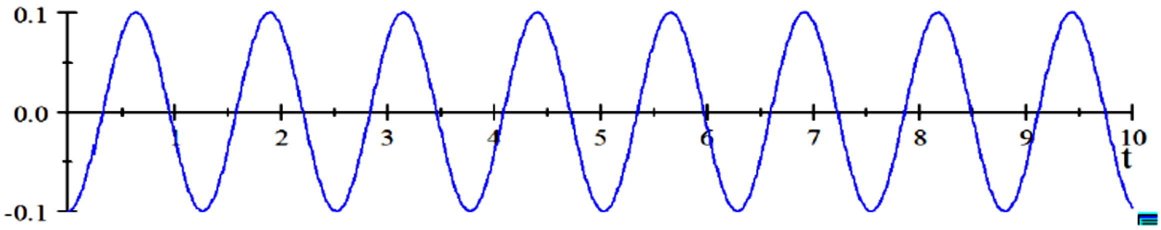
$$\theta_1(t) = -\theta_0 \cos \omega_2 t$$
$$\theta_2(t) = +\theta_0 \cos \omega_2 t$$
$$\omega_2 = \sqrt{\frac{3}{2}\frac{g}{l} + \frac{6ka^2}{ml^2}}$$



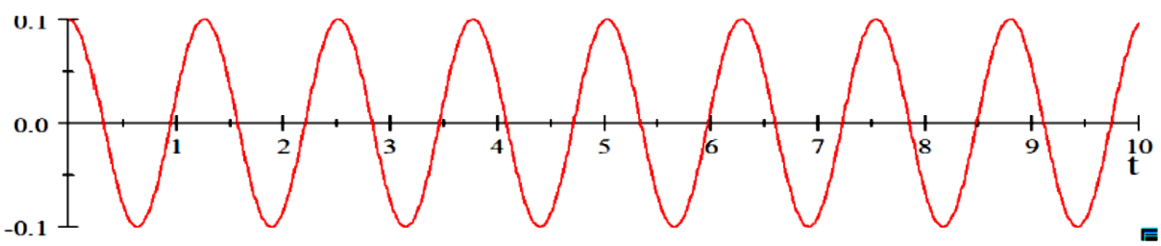
This is again an oscillation at a single frequency, but the coupling between the two pendulums results in a decrease in the period (**equivalent to an increase in frequency**).

$$T_1 = 2\pi\sqrt{\frac{2ml^2}{3\left(mgl + 4ka^2\right)}}$$

$$\theta_1(t) = -0.1\cos(5t)$$



$$\theta_1(t) = +0.1\cos(5t)$$



3. Oscillations with beats.

Initial conditions:

$$\theta_1(0) = 0 \quad \text{et} \quad \theta_2(0) = +\theta_0$$

$$\dot{\theta}_1(t) = \dot{\theta}_2(t) = 0$$

$$\theta_1(0) = A_1 \cos 0 + A_2 \cos 0 = 0$$

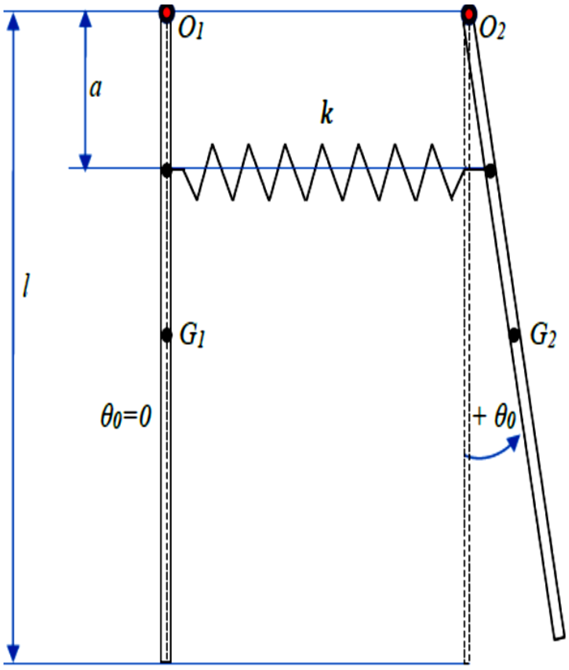
$$\theta_2(0) = A_1 \cos 0 - A_2 \cos 0 = \theta_0$$

$$A_1 = +\theta_0/2$$

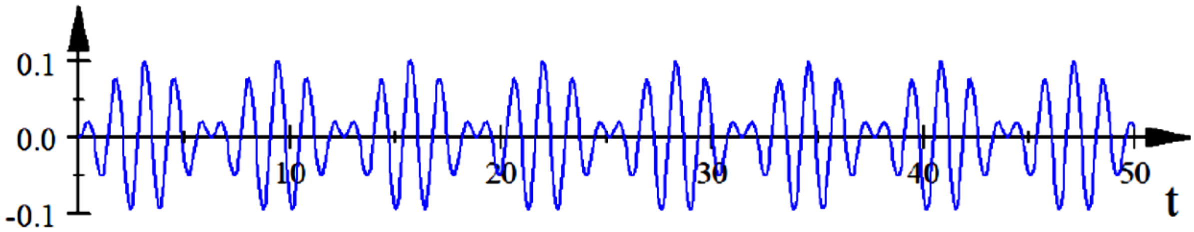
$$A_2 = -\theta_0/2$$

$$\theta_1(t) = \frac{\theta_0}{2}\cos\omega_1 t - \frac{\theta_0}{2}\cos\omega_2 t$$

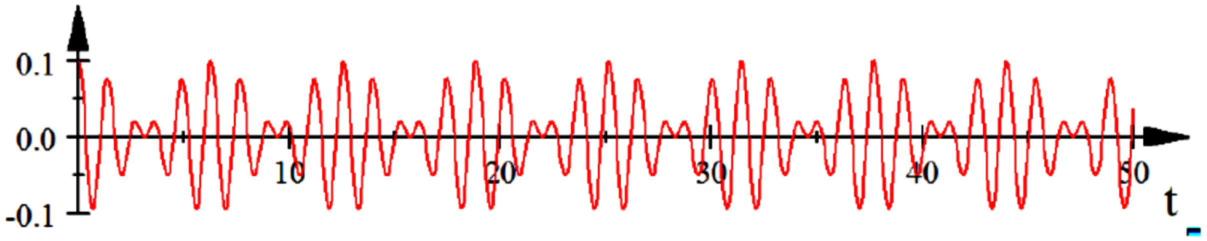
$$\theta_2(t) = \frac{\theta_0}{2}\cos\omega_1 t + \frac{\theta_0}{2}\cos\omega_2 t$$



$$\theta_1(t) = 0.05\cos(3.8t) - 0.05\cos(5t)$$



$$\theta_2(t) = 0.05\cos(3.8t) + 0.05\cos(5t)$$



By using trigonometric relationships, we obtain:

$$\theta_1(t) = \theta_0 \cos(\frac{\omega_2 - \omega_1}{2}t) \times \cos(\frac{\omega_2 + \omega_1}{2}t)$$

$$\theta_2(t) = \theta_0 \sin\left(\frac{\omega_2 - \omega_1}{2}\right)t \sin\left(\frac{\omega_2 + \omega_1}{2}\right)t$$

If the moment of force due to coupling is weak compared to the moment of force due to weight, then $ka^2 \ll mgl$, and we see that ω_1 is close to ω_2, i.e $\omega_2 - \omega_1 \ll \omega_2 + \omega_1$.

We observe that the amplitude of one of the pendulums, varying at the frequency $\left(\frac{\omega_2+\omega_1}{2}\right)$, is modulated by the low frequency $\left(\frac{\omega_2-\omega_1}{2}\right)$. The phase difference of $\left(\frac{\pi}{2}\right)$ between sine and cosine represents the beats between the two pendulums: when one pendulum is at its maximum amplitude, the other is stationary. Mechanical energy gradually transfers from one pendulum to the other during each oscillation via the coupling spring.

- The oscillation period τ is given by: $$\tau = \frac{2\pi}{\left(\frac{\omega_2+\omega_1}{2}\right)} = \frac{2T_1T_2}{T_1+T_2}$$
- The beat period *Tb* (which corresponds to the time between three consecutive stops of the same pendulum) is given by: $$T_b = \frac{2\pi}{\left(\frac{\omega_2-\omega_1}{2}\right)} = \frac{2T_1T_2}{T_1-T_2}$$

We can determine the spring constant *k* by measuring the periods T1 and T2 of the coupled pendulums.

$$k = \frac{mgl}{2a^2}\left(\frac{T_1^2}{T_2^2} - 1\right)$$

Values of frequencies for different positions of the spring.

First case

$$a = \frac{l}{4}$$

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{3}{2}\frac{g}{l} + 6\left(\frac{1}{4}\right)^2\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{3}{2}\frac{g}{l} + \frac{3}{8}\frac{k}{m}}$$

Second case

$$a = \frac{l}{2}$$

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{3}{2}\frac{g}{l} + 6\left(\frac{1}{2}\right)^2\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{3}{2}\frac{g}{l} + \frac{3}{2}\frac{k}{m}}$$

Third case

$$a = \frac{3l}{4}$$

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{3}{2}\frac{g}{l} + 6\left(\frac{3}{4}\right)^2\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{3}{2}\frac{g}{l} + \frac{27}{8}\frac{k}{m}}$$

Required work

- Set up the symmetric and antisymmetric oscillator configurations previously illustrated, taking a=28 cm.

1- Calculate the moment of inertia J_0 of the simple pendulum if:

$\boldsymbol{M} = 147{,}45g$, $\boldsymbol{m} = 1kg$, $\boldsymbol{l} = 96cm$, $L = 100cm$ et $\boldsymbol{g} = 9.8m/s^2$.

$\mathbf{J_0 = J_{Tige} + J_{masse}}$=

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2- Calculate the pulsed extension Δx for m′=50 g and deduce the stiffness constant k.

Δx = ……………………………………………………………………………

k = ……………..……………………………………………………………

3- Using the PASCO Capstone simulation software:

- Calculate for the two oscillations, symmetric and antisymmetric, the periods T1 and T2, and then deduce the averages $\overline{T_1}$ and $\overline{T_2}$ successively.

Measurement	1	2	3	$\mathbf{T_{moyenne}}$
$\mathbf{T_1}$ **(Symmetric)**				$\overline{T_1}$= ………..
$\mathbf{T_2}$ **(Antisymmetric)**				$\overline{T_2}$= ………..

- Calculate the two periods (τ and Tb) for the oscillation with beats, both theoretically and after simulation.

$\boldsymbol{\tau_{th\acute{e}or}}$ = ………………………………………………………………….

$\boldsymbol{\tau_{simul}}$ = ………….**s**

$\mathbf{T_{b\ th\acute{e}or}}$ = ……………………………………………………………

$\mathbf{T_{b\ simul}}$ = ………..**s**

4- Determine the spring constant k of the spring (Dynamic method).

k=……………………………………………………………………………………

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