Contents

Chapter 1

REAL FUNCTIONS OF ONE REAL VARIABLE

1.1 General

Definition domain of a function :

Let $f : E \to \mathbb{R}$ $(E \subset \mathbb{R})$ a function
definition domain of f is

$$D_f = \{x \in E / f(x) \quad \text{exist}\ \}$$

Example:

$$f(x) = \frac{1}{x+1}, \quad D_f = \mathbb{R} - \{1\}.$$

1.1.1 Graph of a real function :

Definition 1.1.1. Graph of a function called the representative curve noted G_f of a function $f : D_f \subset \mathbb{R} \to \mathbb{R}$ is the set of points (x, y).

$$G_f = \Big\{(x, y) \in D_f \subset \mathbb{R} / \quad y = f(x)\Big\}.$$

1.1.2 Direct image , Reciprocal image :

Let $f : E \to \mathbb{R}$ a function $E \subset \mathbb{R}$ and A is a part of E.

* The direct image of A by f is the set denoted by $f(A)$ defined by

$$f(A) = \{f(x) \in \mathbb{R}/x \in A\}.$$

* Let $B \subseteq \mathbb{R}$ the reciprocal image of B by f is the set denoted by $f^{-1}(B)$ defined by

$$f^{-1}(B) = \{x \in E/f(x) \in B\}$$

Example:

$$\begin{aligned} f \ : \ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto x^2 \end{aligned}$$

1. Let $A = [-1, 1]$
   The direct image of A is $f(A) = f([-1, 1]) = [0, 1]$.
   $f([0, +\infty[) = [0, +\infty[$.
   $f(]-\infty, -1]) = [1, +\infty[$.

2. Let $B = \{-2\} \Rightarrow f^{-1}(\{-2\}) = \emptyset$
   $B = \{1\} \Rightarrow f^{-1}(\{1\}) = \{-1, 1\}$.

Remark:
$f^{-1}(\{y\})$ can be empty where containing several elements.
Example:

$$\begin{aligned} f \ : \ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto \sin x \end{aligned}$$

$$f^{-1}(\{1\}) = \{x \in \mathbb{R}, \sin x = 1\} = \left\{\frac{\pi}{2} + 2k\pi\right\} \quad \text{with} \quad k \in \mathbb{Z}.$$

$$f^{-1}(\{0\}) = \{x \in \mathbb{R}, \sin x = 0\} = \{k\pi, k \in \mathbb{Z}\}.$$

$$f^{-1}(\{3\}) = \emptyset.$$

1.2 Even functions, odd functions:

Let $f : E \to \mathbb{R}$ $(E \subset \mathbb{R})$

Definition 1.2.1. A function f defined on an interval E is called even (resp odd) if :

$$\forall x \in I : f(x) = f(-x)(\quad \text{resp} \quad f(x) = -f(-x) \quad \text{or} \quad -f(x) = f(-x))$$

Example:

1. $f(x) = x^2$ and $f(x) = \cos(x)$ are even.

$$f(-x) = (-x)^2 = x^2$$

2. $f(x) = x^3$ and $f(x) = \tan(x)$ are odd.

$*$ When the function f is a even, the points $M(x, f(x))$ and $M'(-x, f(-x))$ are symmetrical about the axis $y'Oy$.
$*$ When the function f is odd, the points $M(x, f(x))$ and $M'(-x, f(-x))$ are symmetrical about the origin.

1.3 Periodic functions:

Definition 1.3.1. $f : E \to \mathbb{R}$ $(E \subset \mathbb{R})$ a function, $T \in \mathbb{R}_+^*$
f is called periodic a period T if :

1. $\forall x \in D_f, x + T \in D_f.$

2. $\forall x \in D_f, f(x+T) = f(x)$.

Example:

1. $f : x \mapsto \sin x, \quad g : x \mapsto \cos x.$ where $T = 2\pi$ the period.

$$f(x+2\pi) = \sin(x+2\pi) = \sin x.$$

$$g(x+2\pi) = \cos(x+2\pi) = \cos x.$$

2. $h : x \mapsto \tan x, T = \pi$ the period
   because : $\tan(x+\pi) = \dfrac{\sin(x+\pi)}{\cos(x+\pi)} = \dfrac{-\sin x}{-\cos(x)} = \tan x.$

1.4 Bounded Functions :

Let $f : E \to \mathbb{R}$ $(E \subset \mathbb{R})$ a function. We say f is :

1. Bounded above if : $\exists M \in \mathbb{R}, \forall x \in E; f(x) \leq M$.
2. Bounded below $\exists m \in \mathbb{R}, \forall x \in E; f(x) \geq m$.
3. Bounded if : it is bounded above and bounded below.
   We can also write :

$$\exists M \in \mathbb{R}, \forall x \in E; |f(x)| \leq M.$$

Example:

1. $\forall x \in \mathbb{R} : |\sin x| \leq 1$ and $|\cos x| \leq 1$.
   So : $x \mapsto \sin x$ et $x \mapsto \cos x$ are bounded functions.
2. $f(x) = \dfrac{1}{x^2+1}, \forall x \in \mathbb{R} : x^2 + 1 \geq 1 \Rightarrow 0 < \dfrac{1}{x^2+1} \leq 1.$
   so f is bounded.

1.5 Monotone functions:

$f : E \to \mathbb{R}$ ($E \subset \mathbb{R}$) a function.
f is called monotone increasing (resp decreasing) in E if :

$$\forall x, y \in E : x \leq y \Rightarrow f(x) \leq f(y) (\text{resp} f(x) \geq f(y))$$

1.6 Local maximum , local minimum

Let $f : E \to \mathbb{R}$ a function

1. We say f admits a local Maximum in $x_0 \in E$ if $f(x_0)$ is the maximum of the part $f(E)$ so that

   $$f(E) = \left\{ f(x)/x \in E = D_f \right\}$$

   (M is the maximum of f on $E \Leftrightarrow f(x) \leq M, \forall x \in E = D_f$.)

2. m is the minimum of f on $E \Leftrightarrow f(x) \geq m, \forall x \in E = D_f$.)

3. We say f admits a local maximum in $x_0 \in E = D_f$, if an open interval exists I containing x_0 so that $f(x_0)$ the maximum of $f(E \bigcap I)$.

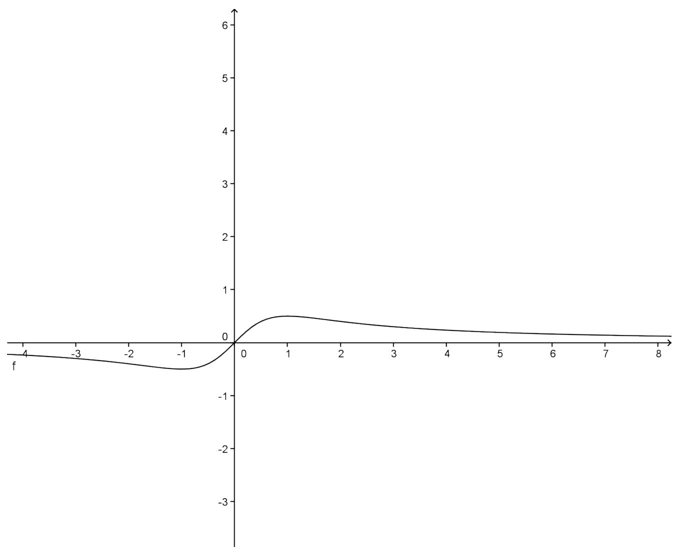
4. We say f admits a local minimum in $x_0 \in E = D_f$, if an open interval exists I containing x_0 so that $f(x_0)$ the minimum of $f(E \bigcap I)$.

5. An extremum (local) is a maximum (local) or a minimum (local).

Remark:
A function bounded always accept an upper bound and a lower bound but not necessarily a maximum and a minimum.
Example:

$$1) \quad f \ : \ \mathbb{R} \to \mathbb{R}$$
$$x \mapsto f(x) = \frac{x}{x^2+1}$$

on the interval $I =]-2,2[$ open contained in $D_f = \mathbb{R}$
f admits a local minimum $x_0 = f(-1) = \frac{-1}{2}$.
and a maximum local $A = f(1) = \frac{1}{2}$.



2) $f(x) = \sqrt{1-x^2}$, $\qquad D_f = [-1,1]$.
f admits in 1 and -1 local minimums and a maximum strictly en 0.

1.7 Limit of a function:

Definition 1.7.1. A part $V \subseteq \mathbb{R}$ is a neighborhood of
$x_0 \in \mathbb{R}$ if an open interval exists I so that $x_0 \in I \subset V$.

Example:
$V = [1,5[$ is a neighborhood of $x_0 = 2$
Because : for example $2 \in]1,5[\subset V$.

Definition 1.7.2. Let be V a neighborhood of $x_0 \in \mathbb{R}$ and $f : V \to \mathbb{R}$ it's a function. We say f has a limit $l (l \in \mathbb{R})$ in x_0 if:

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x \in V : 0 < |x - x_0| < \eta \Rightarrow |f(x) - l| < \varepsilon$$

and , we write : $\lim\limits_{x \to x_0} f(x) = l$

Example:

1. $f(x) = 5x - 4, x_0 = 1$
   $\lim\limits_{x \to 1} f(x) = 1.$
   Indeed, let $\varepsilon > 0$, we're looking for $\eta > 0$.

$$\begin{aligned} |f(x) - 1| < \varepsilon &\Leftrightarrow |5x - 5| < \varepsilon \\ &\Leftrightarrow 5|x - 1| < \varepsilon \\ &\Leftrightarrow |x - 1| < \frac{\varepsilon}{5} \end{aligned}$$

   Just take $\eta = \dfrac{\varepsilon}{5}$

2. $f(x) = \dfrac{\sin x}{x}, \quad x_0 = 0.$
   $\lim\limits_{x \to 0} f(x) = 1.$ Note that $f(0) \nexists$

3.
$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{si} \quad x \neq 0. \\ 2 & \text{si} \quad x = 0. \end{cases} \qquad D_f = \mathbb{R}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

Remark:

If $\lim_{x\to x_0} f(x)$ exists, so $f(x_0)$ can exist or not, even if : $f(x_0)$ exists, it may be different from $\lim_{x\to x_0} f(x)$.

1.8 Left and Right-Hand Limits :

Definition 1.8.1. We say f has a limit on the right (resp a left) at the point x_0 if :
$\forall \varepsilon > 0, \exists \eta > 0, \forall x, x_0 < x < x_0 + \eta \Rightarrow |f(x) - l| < \varepsilon$(resp $\quad x_0 - \eta < x < x_0) \Rightarrow |f(x) - l| < \varepsilon$.
And, we write : $\lim_{x\to^> x_0} f(x)$(resp $\quad \lim_{x\to^< x_0} f(x)$)

Example:

1. $f(x) = E(x), \qquad x_0 = 2.$

$$\lim_{x\to^> 2} f(x) = 2, \quad \lim_{x\to^< 2} f(x) = 1.$$

   So : $\lim_{x\to 2} f(x)$ does not exist.
   Note that $f(2) = E(2) = 2$.

2. $f(x) = \dfrac{|x|}{x}, \quad x_0 = 0.$
   $\lim_{x\to^> 0} f(x) = 1, \quad \lim_{x\to^< 0} f(x) = -1.$
   $f(0)$ does not exist.

Infinite limit:

(*) $\lim_{x\to x_0} f(x) = +\infty \Leftrightarrow \forall A > 0, \exists \eta, \forall x, 0 < |x - x_0| < \eta \Rightarrow f(x) > A.$

(*) $\lim_{x\to x_0} f(x) = -\infty \Leftrightarrow \forall A > 0, \exists \eta, \forall x, 0 < |x - x_0| < \eta \Rightarrow f(x) < -A.$
if: $x_0 = +\infty$ or $x_0 = -\infty$, we will pose :

(*) $\lim_{x\to +\infty} f(x) = +\infty \Leftrightarrow \forall A > 0, \exists B > 0$ such that $x > B \Rightarrow f(x) > A.$

(*) $\lim_{x\to -\infty} f(x) = +\infty \Leftrightarrow \forall A > 0, \exists B > 0$ such that $x < -B \Rightarrow f(x) > A.$

(*) $\lim_{x\to+\infty} f(x) = -\infty \Leftrightarrow \forall A > 0, \exists B > 0$ such that $x > B \Rightarrow f(x) < -A$.

(*) $\lim_{x\to-\infty} f(x) = -\infty \Leftrightarrow \forall A > 0, \exists B > 0$ such that $x < -B \Rightarrow f(x) < -A$.

1.9 Limits theorem

Relationship between the limits of functions and limits of sequences.

Theorem 1:

Let $f : [a,b] \to \mathbb{R}$ a function
the following two properties are equivalent

(i) $\lim_{x\to x_0} f(x) = l$

(ii) Whatever a sequence $(x_n), x_n \in [a,b]$, such that $x_n \neq x_0$ and $\lim_{x\to+\infty} x_n = x_0$. we have: $f(x_n) = l$.(l complete or infinite)

Proof. i) $\Rightarrow$ ii)
we have:
$\lim_{x\to x_0} f(x) = l \Leftrightarrow \forall\varepsilon > 0, \exists\eta > 0, \forall x \in [a,b], 0 < |x - x_0| < \eta \Rightarrow |f(x) - l| < \varepsilon$.
and $\lim_{x\to+\infty} x_n = x_0 \Leftrightarrow \forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N \Rightarrow |x_n - x_0| < \varepsilon$.
We take $\varepsilon = \eta$, we find:
$\forall n \geq N \Rightarrow |x_n - x_0| < \eta \quad (\varepsilon = \eta) \Rightarrow 0 < |x_n - x_0| < \eta \quad (\text{car} \quad x_n \neq x_0)$
so: $|f(x_n) - l| < \varepsilon$.
Which shows that $\lim_{n\to+\infty} f(x_n) = l$
ii)$\Rightarrow$ i) Exercise. □

Example:
$f(x) = \sin(\frac{1}{x}), \qquad \lim_{x\to 0} f(x) = ?$

$$x_n = \frac{1}{\frac{\pi}{2} + 2n\pi}, \qquad \lim_{n\to+\infty} x_n = 0.$$

$$\lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} \sin\left(\frac{1}{\frac{1}{\frac{\pi}{2}+2n\pi}}\right) = \lim_{n\to+\infty} \sin\left(\frac{\pi}{2}+2n\pi\right) = 1.$$

$$y_n = \frac{1}{n\pi}, \lim_{n\to+\infty} f(y_n) = \lim_{n\to+\infty} \sin(n\pi) = 0.$$

We deduce that $\lim_{x\to 0} \sin\frac{1}{x}$ doesn't exist.

theorem 2: theorem for framing (squeez)

If: $f \leq g \leq h$ and if $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = l$.
so

$$\lim_{x\to x_0} g(x) = l$$

1.10 Operations on limits:

Let f and g two real functions such that:

$$\lim_{x\to x_0} f(x) = l_1, \lim_{x\to x_0} g(x) = l_2, \quad \text{and} \quad \lambda \in \mathbb{R},$$

then:

1. $\lim_{x\to x_0} \lambda f(x) = \lambda l_1$.
2. $\lim_{x\to x_0} (f+g)(x) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = l_1 + l_2$.
3. $\lim_{x\to x_0} (f\times g)(x) = \lim_{x\to x_0} f(x) \times \lim_{x\to x_0} g(x) = l_1 \times l_2$.
4. $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{l_1}{l_2}, \quad l_2 \neq 0$
5. $\lim_{x\to x_0} |f(x)| = |l|$

6. Let be x_0, y_0 and l real, f and g two functions
   then :
$$\begin{cases} \lim_{x \to x_0} f(x) = y_0 \\ \lim_{x \to y_0} g(x) = l. \quad \text{so} \quad \lim_{x \to x_0} g \circ f(x) = l \end{cases}$$

Proposition 1.10.1. 1. If: $f \leq g$ and $\lim_{x \to x_0} f(x) = l_1$ and $\lim_{x \to x_0} g(x) = l_2$ so $l_1 < l_2$

2. If: $f \leq g$, $\lim_{x \to x_0} f(x) = +\infty$ so $\lim_{x \to x_0} g(x) = +\infty$

theorem 1

If f admits a limit l then f is bounded

Proof. $|f(x) - l| < \varepsilon \Rightarrow -\varepsilon < f(x) - l < \varepsilon$
so: $l - \varepsilon < f(x) < l + \varepsilon$
so f is bounded. □

theorem 2

If f is bounded and $\lim_{x \to x_0} g(x) = 0$
then : $\lim_{x \to x_0} f(x) \times g(x) = 0$

Relationship between sequence and function:
If f admits a limit l at $x_0(\lim_{x \to x_0} f(x) = l)$.
So for all sequence (x_n) which converges to x_0 We have : $\lim_{n \to +\infty} f(x_n) = l$.

1.11 Continuous functions

1.11.1 Continuity at a point:

Let a function $f : I \to \mathbb{R}$ (I being the interval of $\mathbb{R}$),
We say f continue at $x_0 \in I$ if and only if:

$$\lim_{x \to x_0} f(x) = f(x_0) \quad \text{i.e:}$$
$$\forall \varepsilon > 0, \exists \eta > 0, \forall x \in I : [0 < |x - x_0| < \eta \Rightarrow |f(x) - f(x_0)| < \varepsilon]$$

1.11.2 Right-continuous and left-continuous:

(*) We say f is continuous to the right of $x_0$if and only if

$$\lim_{x \to^> x_0} f(x) = f(x_0)$$

(*) We say f is continuous to the left of x_0 if and only if

$$\lim_{x \to^< x_0} f(x) = f(x_0)$$

Remark: f continue in $x_0 \Leftrightarrow \lim_{x \to^< x_0} f(x) = f(x_0) = \lim_{x \to^< x_0} f(x) = f(x_0)$

Example:

1. $f(x) = |x|, \quad x_0 = 0$

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$\lim_{x \to^> 0} f(x) = 0 = f(0) = \lim_{x \to_< x_0} f(x) =$
so f is continuous at x_0.

2. $f(x) = E(x), \quad x_0 = n \quad (n \in \mathbb{R})$
we have: $E(n) = n, \lim_{x \to^> n} f(x) = n, \lim_{x \to^< n} f(x) = n - 1.$

$$\lim_{x \to^> n} f(x) \neq \lim_{x \to^< n} f(x)$$

so f is discontinuous at $x_0 = n \quad (n \in \mathbb{R})$

⋇ We can also express the continuity using the relationship between the limit of a function and convergent sequences (see the previous lesson)

(f continue in $x_0 \in I \Leftrightarrow$ whatever sequences $(x_n) \in I$)

If $\lim\limits_{n\to+\infty} x_n = x_0$ so $\lim\limits_{n\to+\infty} f(x_n) = f(x_0)$

1.12 Operations on continuous functions:

Theoreme 1:

If $f, g : I \to \mathbb{R}$ are continuous functions at $x_0 \in I$ so :

$(*)$ $\alpha.f(\alpha \in \mathbb{R}), f+g, f \times g$ and $\dfrac{f}{g}$ $(g(x_0) \neq 0)$ are continuous at x_0.

$(*)$ Si f is continuous at x_0 and g continue in $f(x_0)$ so : $g \circ f$ is continuous at x_0 $\left(\lim\limits_{x\to x_0}(g\circ f)(x) = g\circ f(x_0)\right)$.

Continuity of the composite function

Theoreme 2:

Let $f : I \to I$ and $g : I \to \mathbb{R}$ two functions

If f is continuous at x_0 and g continue in y_0 with $y_0 = f(x_0)$.

so: $g \circ f$ is continuous at x_0

Proof. g continue in $y_0 = f(x_0) \Leftrightarrow$

$\forall\varepsilon > 0, \exists\eta > 0, \forall y \in I : 0 < \left|y - y_0\right| < \eta \Rightarrow \left|g(y) - g(y_0)\right| < \varepsilon$

As f is continuous at x_0 :

$$\begin{aligned}\exists\eta > 0, \forall x \in I : [0 < |x - x_0| < \eta \Rightarrow |f(x) - f(x_0)| < \varepsilon < \eta]\\ \Rightarrow |g(f(x)) - g(f(x_0))| < \varepsilon.\\ \Rightarrow |g\circ f(x) - g\circ f(x_0)| < \varepsilon.\end{aligned}$$

Which proves that $(g \circ f)$ is continuous at x_0 □

1.13 Extension by continuity

Definition :

Let f a function defined and continuous on $I/x_0\,(I - x_0)$

Suppose that $\lim\limits_{x \to x_0} f(x)$ exists.

So : a function $\widetilde{f}$ defined on I

$$\widetilde{f}(x) = \begin{cases} f(x) & if \quad x \neq x_0. \\ \lim\limits_{x \to x_0} f(x) & if \quad x = x_0. \end{cases}$$

and known as the continuity extension of f en x_0.

Example:

$f(x) = \dfrac{\sin x}{x}, \quad D_f = \mathbb{R} - \{0\}$

$\lim\limits_{x \to 0} f(x) = 1$ so $\widetilde{f}$ defined by :

$$\widetilde{f}(x) = \begin{cases} f(x) & if \quad x \neq 0. \\ \lim\limits_{x \to 0} f(x) & if \quad x = 0. \end{cases}$$

$$\widetilde{f}(x) = \begin{cases} \dfrac{\sin x}{x} & if \quad x \neq 0. \\ 1 \quad if \quad x = 0. \end{cases}$$

$\widetilde{f}$ is continuous on $\mathbb{R}$.

$\widetilde{f}$ is the function extended by continuity of f at 0.

1.14 Continuous function on an interval

Definition 1.14.1. The function f defined on interval $I \subseteq \mathbb{R}$ is continuous on I provided f is continuous at every point of I.

Remark:
Polynomial functions and rationals are continuous on their domain of definition.
Example:

$f(x) = x^2 - 2x + 1$ continues on $\mathbb{R}$.
$g(x) = \dfrac{x^3 - 4}{x}$ continues on $\mathbb{R} - \{0\}$.
$h(x) = \sqrt{x}$ continues on $\mathbb{R}^+$

1.15 Theorem for continuous functions on a closed interval :

Let f a function continuous on the closed interval $[a, b]$, so:

1. f is bounded (i.e, $\forall x \in [a, b], |f(x)| \leq M$)
2. f reaches its upper bound and its lower bound that is:

$$\exists x_1 \in [a, b], f(x_1) = \sup f(x), \quad x \in [a, b].$$

$$\exists x_2 \in [a, b], f(x_2) = \sup f(x), \quad x \in [a, b].$$

Proof. 1) Suppose that f continuous and unbounded on $[a, b]$
So : $\forall n \in \mathbb{N}, \exists x_n \in [a, b], |f(x_n)| > n.$
the sequence (x_n) is bounded $\Rightarrow$ (x_n) admits an extracted sequence $(x_{\phi(n)})$ converges.

$\lim_{n\to+\infty} x_{\phi(n)} = \overline{x} \in [a,b]$ because $a < x_{\phi(n)} < b$
We have : $\lim_{n\to+\infty} \left|f(x_{\phi(n)})\right| = |f(\overline{x})| < +\infty \ldots (1)$
as f is continuous on $[a,b]$ and $x_{\phi(n)} \to \overline{x}$
So : $\lim_{n\to+\infty} \left|f(x_{\phi(n)})\right| = |f(\overline{x})| < +\infty.$
Contradiction with (1).

So f is bounded .

2) $\sup f(x)$ et $\inf f(x)$ exists.

a) Suppose that $\forall x \in [a,b], f(x) < \sup f(x) = M$
Let's put : $g(x) = \dfrac{1}{M - f(x)}$, $\quad g$ continues on $[a,b]$.
According to g is bounded.

$$\Rightarrow \exists c \in \mathbb{R} : |g(x)| < c, \forall x \in [a,b]$$

$\Rightarrow \dfrac{1}{M - f(x)} \leq c \Rightarrow M - f(x) \geq \dfrac{1}{c}$, so:$f(x) \leq M - \dfrac{1}{c}$.
Contradiction with $\sup f = M$.

so : $\exists x_1 \in [a,b] : f(x_1) = \sup(f(x)), x \in [a,b]$. □

Another way:

Let f a function from $[a,b]$ at $\mathbb{R}$. We say f is continuous on $[a,b]$ if:

1. f is continuous on interval $]a,b[$.
2. f is right-continuous at a.
3. f is left-continuous at b.

1.16 Discontinuity of first and second species

(1) f has a discontinuity of the first kind in x_0 if: f is not continuous in x_0 and admits a right-limit and a left- limit at x_0.

(2) f has a discontinuity of the second kind if: f is not continuous in x_0 and discontinuity is not the first species.

Example:

1)

$$f(x) = \begin{cases} \frac{2\sqrt{x^2}}{x} & \text{if} \quad x \neq 0. \\ 2 & \text{if} \quad x = 0. \end{cases}$$

$$f(x) = \frac{2\sqrt{x^2}}{x}, \quad \text{if} \quad x \in]-\infty, 0[\cup]0, +\infty[, f(0) = 2$$

(⋇) $\lim\limits_{x \to^{>} 0} f(x) = \lim\limits_{x \to^{>} 0} \frac{2\sqrt{x^2}}{x} = \lim\limits_{x \to^{>} 0} \frac{2|x|}{x} = \lim\limits_{x \to^{>} 0} \frac{2x}{x} = 2.$

(⋇) $\lim\limits_{x \to^{<} 0} f(x) = \lim\limits_{x \to^{<} 0} \frac{2\sqrt{x^2}}{x} = \lim\limits_{x \to^{<} 0} \frac{2|x|}{x} = \lim\limits_{x \to^{<} 0} \frac{2(-x)}{x} = -2.$

so: $\lim\limits_{x \to 0^+} f(x) \neq \lim\limits_{x \to 0^-} f(x)$ so: f has a discontinuity of the first kind.

2) $g(x) = \sin \frac{1}{x}, \quad D_g = \mathbb{R} - \{0\}.$

g is not continuous in $x_0 = 0$.

So : g has a discontinuity of the second kind.

1.17 The intermediate value theorem

Theoreme 1:

Suppose f is a continuous function on the interval $[a, b]$ if:$f(a).f(b) < 0$.

So: $\exists c \in]a, b[: f(c) = 0.$

Theoreme 2:

Let $f : I \to \mathbb{R}$ is a continuous function on I any interval of $\mathbb{R}$.

Let $a < b$. two elements of I, takes every value l that is between $f(a)$ and $f(b)$

So: $\exists c \in]a, b[: f(c) = l.$

Proof. Suppose that $f(a) < f(b)$ so $f(a) < l < f(b)$.
Let $g(x) = f(x) - l$
g is a continuous on $[a,b]$.

$$\begin{cases} g(a) = f(a) - l < 0 \\ g(b) = f(b) - l > 0 \end{cases} \quad \Rightarrow f(a) - f(b) < 0$$

According to the previous theorem

$$\exists c \in]a,b[: g(c) = 0 \Rightarrow g(c) = f(c) - l = 0 \Rightarrow f(c) = l.$$

□

Example:
$f(x) = x^5 - 3x + 1$.
Show that $\exists c \in]0,1[, f(c) = 0$
f is a continuous on $[0,1]$, and we have :$f(0) \times f(1) = -1 < 0$.
So :$\exists c \in]0,1[: f(c) = 0$

1.18 Uniform continuity

Definition:

A function f defined on interval I is uniformly continuous (U.C) on I if:

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x,y \in I : \left[\left| x - y \right| < \eta, \left| f(x) - f(y) \right| < \varepsilon \right]$$

This means:

$$\forall x_n, y_n \in I : \left| x_n - y_n \right| \to 0 \Rightarrow \left| f(x_n) - f(y_n) \right| \to 0.$$

Example:

$$f(x) = \frac{1}{x}, I = [1, +\infty[.$$

Let $\varepsilon > 0$ and $x, y \in [1, +\infty[$.

$$\left|f(x) - f(y)\right| = \left|\frac{1}{x} - \frac{1}{x}\right| = \left|\frac{y-x}{xy}\right|.$$

$$x \geq 1 \quad \text{et} \quad y \geq 1 \Rightarrow xy \geq 1 \Rightarrow \frac{1}{xy} \leq 1.$$

$$\text{so:} \quad \frac{\left|y-x\right|}{xy} = \frac{\left|x-y\right|}{xy} < \left|x-y\right|.$$

$$\text{so:} \quad \left|f(x) - f(y)\right| < \left|x-y\right|.$$

So just take $\eta = \varepsilon, \left(\left|x-y\right| < \eta \Rightarrow \left|f(x) - f(y)\right| < \varepsilon\right)$
So f is uniformly continuous (U.C) on $[1, +\infty[$.

Remark:

(1) The uniform continuity is a property of f on interval while continuity can be defined at a point.

(2) If f is U.C on I so f is a continuous on I.

(3) In general if f is a continuous on I so f is not necessarily U.C on I.

Example:

$$f(x) = \frac{1}{x}, \qquad I =]0, 1].$$

f is continuous on $]0, 1]$.
P:(f is U.C)$\Leftrightarrow \forall x_n, y_n \in]0, 1], \left|x_n - y_n\right| \to 0 \Rightarrow \left|f(x_n) - f(y_n)\right| \to 0$.
$\overline{P}$:(f is not U.C)$\Leftrightarrow \exists x_n, y_n \in]0, 1], \left|x_n - y_n\right| \to 0 \wedge \left|f(x_n) - f(y_n)\right| \nrightarrow 0$.

$$\overline{A \Rightarrow B} \Leftrightarrow A \wedge \overline{B}, \quad A \Rightarrow B \Leftrightarrow \overline{A} \vee B.$$

We take: $x_n = \dfrac{1}{n}, \qquad y_n = \dfrac{1}{n+1}, n \neq 0.$

$$\left|x_n - y_n\right| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} \to 0.$$

$$\text{But} \quad \left|f(x_n) - f(y_n)\right| = |n-(n+1)| = 1 \nrightarrow 0.$$

So f is not U.C on $]0,1]$

1.19 Lipschitz Function:

Definition:

Let I an interval on $\mathbb{R}$, a function $f : I \to \mathbb{R}$ is called Lipschitz if a constant exists $k \geq 0$ such that :

$$\forall x, y \in I : \left|f(x) - f(y)\right| \leq k\left|x - y\right|.$$

According to the previous definition:
f Lipschitz on $I \Rightarrow f$ is U.C on I.
Indeed : $\left|f(x) - f(y)\right| \leq k\left|x-y\right| < (k+1)\left|x-y\right|$.
So : just take the $\eta = \dfrac{\varepsilon}{k+1}$.

Example:

$$f(x) = x^2, \qquad I = [0,1].$$

$\forall x, y \in [0,1], \left|f(x) - f(y)\right| = \left|x^2 - y^2\right| = \left|(x+y)(x-y)\right| = \left|x+y\right|\left|x-y\right| = (x+y)\left|x-y\right| \leq 2\left|x-y\right|$.

$$\begin{cases} x \leq 1 \\ y \leq 1 \end{cases} \Rightarrow (x+y) \leq 2$$

So: f is Lipschitz $\Rightarrow f$ is U.C on $[0,1]$.

1.20 Heine's theorem:

A function continuous on a closed interval $[a,b]$ is uniformly continuous on this interval.

Example:
$f(x)=\frac{1}{x}$, f is a continuous on $\left[\frac{1}{2},1\right] \Rightarrow f$ is U.C on $\left[\frac{1}{2},1\right]$.

1.21 Continuous reciprocal function

Theorem:
If f a function continuous and strictly monotone on $[a,b]$ so f is a bijection of $[a,b]$ in $f([a,b])$.

Definition

Let f a bijection, so f admits a reciprocal function f^{-1} continuous and strictly monotone on $f([a,b])$ and with the same sense of monotony of f.

$$\begin{aligned} f^{-1} \;:\; & f([a,b]) \to [a,b] \\ & x \mapsto f^{-1}(x). \end{aligned}$$

Example:

$$\begin{aligned} f \;:\; & \mathbb{R} \to \mathbb{R} \\ & x \mapsto f(x) = 4x-1. \end{aligned}$$

$$\begin{aligned} f^{-1} \ : \ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto f^{-1}(x). \end{aligned}$$

We have $f(x) = y = 4x - 1 \Rightarrow x = \frac{y+1}{4}$, so $x = \frac{y}{4} + \frac{1}{4} = f^{-1}(y)$.
so : $f^{-1}(x) = \frac{1}{4}x + \frac{1}{4}$.

1.22 Order of a variable-equivalence (Landau's notation):

1.22.1 Negligible functions:

Definition:

Let f, g two functions defined on the same interval I in a neighborhood of the point x_0 (left and right of $x_0 \in \overline{\mathbb{R}}$ with $\overline{\mathbb{R}} = [-\infty, +\infty]$).
We say f is negligible compared to g in the neighbourhood of x_0 if a function exists ε defined on I with $f = \varepsilon \times g$ et $\lim_{x \to x_0} \varepsilon(x) = 0$.
It is symbolized by : $f(x) = O_{x_0} g(x)$.
We call $f = O(g)$ Landau notation.

Remark:
If the function g does not cancel out in the neighbourhood of x_0.
So : $f = O(g) \Leftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$.
Example:
1)

$$x = O_{+\infty} x^2, \exists \varepsilon \quad \text{whith} \quad \varepsilon(x) = \frac{1}{x}.$$

$$x = \frac{1}{x} x^2, \quad \varepsilon = \frac{f}{g}, \quad \text{such that} \quad \lim_{x \to +\infty} \varepsilon(x) = 0.$$

$$x = O_{+\infty}(x^2) \Leftrightarrow \lim_{x \to +\infty} \frac{x}{x^2} = 0$$

2) $x + 1 = O_{+\infty}(x)^2$.

$x = O_{+\infty}(e^x)$.
$x^\alpha = O_{+\infty}e^{\beta x}(\alpha, \beta > 0)$.

1.23 Equivalent functions

We say f is equivalent to g in the neighbourhood of x_0 if and only if $(f - g)$ is negligible compared to g.
We write:

$$f \sim_{x_0} g \Leftrightarrow f - g = O_{x_0} g$$

$$\sin x \sim x, f - g = O_{x_0} f \begin{cases} f - g = \varepsilon_1 g. \\ f - g = \varepsilon_2 g. \end{cases} \quad g \neq 0, \lim_{x \to x_0} \frac{f - g}{g} = 0.$$

$$f \neq 0, \lim_{x \to x_0} \frac{f}{g} - \lim_{x \to x_0} \frac{g}{g}$$

$$\text{such that} \quad f \neq 0 \quad \text{and} \quad g \neq 0 \quad \text{on} \quad V - \{x_0\}.$$

$$\text{So:} \quad f \sim g \Leftrightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$$

Example:

$$\sin x \sim x.$$

$$\cos x \sim 1 - \frac{x^2}{2}.$$

$$\ln(1 + x) \sim x.$$

$$e^x - 1 \sim x.$$